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Numerical Solution of Integral Equations Using Homotopy Perturbation Method and Series Solution Method

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ABSTRACT

The aim of this research paper is to develop numerical methods of solving linear Volterra integral equations using Homotopy Perturbation Method (HPM) and Series Solution Method (SSM). Most Integral equations are difficult to solve analytically, hence the need for a more accurate and reliably numerical method. In the the Homotopy Perturbation approach, the modelled problem is used to construct a homotopy and a perturbation expansion in terms of power series was assumed while in the series solution method, a power series solution was assumed and the assumed solution was substituted into the modelled problem by comparing the coefficient of like terms to obtain the unknown constants. The results of the two methods were compared and numerical examples were used to establish the simplicity, reliability and efficiency of the methods. The result shows that the methods are accurate and outperforms other methods existing in literature.

Keywords: Volterra, Perturbation, Integral equation, Homotopy, Power Series

MSC 2020 Subject classification: Primary 45A05; Secondary 65R20

INTRODUCTION

Many problems in mathematical physics can be solved using integral equations. In recent years, there has been a growing interest in the volterra integral equations which was used in modelling several phenomena in physics, engineering and predator-prey model in biological sciences.(Zarnan, 2018).

Several methods have been developed for solving linear Volterra integral equations which includes Power Series Collocation Method (Ganiyu et al., 2024), Least-Square Method (Al-Humedi and Shoushan, 2021), Adomian Decomposition Method (Khan and Bakoda, 2013) and many more. Most Integral equations are difficult to solve analytically, hence the need for a more accurate and reliably numerical method which can be able to solve Volterra integral equations will be a powerful tool and a timely intervention.

In this research work, linear volterra integral equation of the second kind of the form:

$$y(x) + \alpha \int_0^x k(x,t)y(t)dt = f(x) \quad (1)$$

Where $k(x,t)$ is the volterra integral kernel, α is a known parameter given, $f(x)$ is a known function, and $y(x)$ is the unknown function to be determined.

MATERIALS AND METHODS

In this section, the approximation approach for the numerical solution of linear Volterra integral equations of the second kind is implemented.

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Case 1 to Method 1: Homotopy Perturbation Method

This method introduces a homotopy parameter that transforms the integral into an iterative solution. To illustrate the Homotopy Perturbation Method, consideration is given to equation (1) as

$$L(y) = y(x) + \alpha \int_0^x k(x,t)y(t)dt - f(x) = 0 \quad (2)$$

Using the homotopy technique, we can define homotopy $H(y,p)$ by

$$H(y,p) = (1-p)F(y) + PL(y) \quad (3)$$

Where $F(y)$ is a functional operator with known solution y_0, and equation (3) gives;

$$H(y,0) = F(y); \quad H(y,1) = L(y) \quad (4)$$

And changing the parameter p from $0\ to\ 1$, we continuously trace an implicitly defined curve $H(u,p)$ from a starting point $H(y,0)\ to\ a\ solution\ function\ H(y,1)$. Applying the perturbation technique [5], due to the fact that $0 \le p \le 1$, p can be considered as a small parameter. We assumed that the solution of equation (3) can be expressed as a series as follows;

$$y = \sum_{n=0}^{\infty} p^n y_n = y_0 + py_1 + p^2y_2 + p^3y_3 + \dots \quad (5)$$

As $p \to 1$, equation (3) tends to equation (2) and equation (5) gives the approximate solution of equation (3);

$$y(x) = lim_{p\to 1} y = \sum_{n=0}^{\infty} y_n = y_0 + y_1 + y_2 + y_3 + \dots \quad (6)$$

Applying the homotopy technique (3) to equation (1) gives

$$y(x) = f(x) - \alpha \int_0^x (1-p)k(x,t)y(t)_e + pk(x,t)y(t)dt \quad (7)$$

Putting equation (5) into equation (7) gives;

$$y_0 + py_1 + p^2y_2 + \dots = f(x) - \alpha \int_0^x (1-p)k(x,t)y(t)_e + pk(x,t)(y_0 + py_1 + p^2y_2 + \dots)dt \quad (8)$$

Equating like powers of p and solving gives

$$y(x) = \sum_{n=0}^{\infty} y_n = y_0 + y_1 + y_2 + y_3 + \dots \quad (9)$$

Case 2 to Method 2: Series Solution Method

This method represents the solution of the integral as an infinite series, which helps in evaluation of the approximate solution. To illustrate the Series Solution Method (SSM), let the approximate solution $y(x)$ be given as;

$$y(x) = \sum_{n=0}^{N} a_n x^n \ and\ y(t) = \sum_{n=0}^{N} a_n t^n, \qquad n = 0(1)N \quad (10)$$

$$\sum_{n=0}^{N} c_n x^n + \alpha \int_0^x k(x,t) \sum_{i=0}^{N} c_n t^n \, dt = f(x) \quad (11)$$

$$c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \alpha \int_0^x k(x,t)(c_0 + c_1 t + c_2 t^2 + \dots + c_n t^n) dt = f(x) \quad (12)$$

$$c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n = f(x) - \alpha \int_0^x k(x,t)(c_0 + c_1 t + c_2 t^2 + \dots + c_n t^n) dt \quad (13)$$

Comparing the coefficients of like powers of x and substituting the values of $c_i's$ for $i = 0(1)N$ in

$y(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n$ gives the solution of equation (1).

NUMERICAL EXAMPLES

In this section, we test the accuracy, reliability and efficiency of the numerical technique on some numerical examples. Let $y_N(x)$ and $y(x)$ be the approximate and exact solutions respectively, then the error is given by;

$Error_N = |y_N(x) - y(x)|$

EXAMPLE 1:

Considering Volterra integral equation

$$y(x) - \int_0^x (t - x) y(t) dt = x \quad (14)$$

Exact solution $y(x) = \sin(x)$

Source: Zarnan (2018).

SOLUTION 1:

CASE 1:

Applying the homotopy on equation (14) gives;

$$y(x) = x + \int_0^x (1 - p)(t - x) sin(t) + p(t - x) y(t) dt \quad (15)$$

Substituting equation (5) into equation (15) and collecting like powers of p gives

$$p^0: y_0 = x + \int_0^x (t - x) sin(t) dt \Longrightarrow y_0 = \sin(x)$$

$$p^1: y_1 =- \int_0^x ((t - x) sin(t) + (t - x)\sin(t))dt \Longrightarrow y_1 = 0$$

$$p^2: y_2 = \int_0^x (t - x)(0) dt \Longrightarrow y_2 = 0$$

$$p^3: y_3 = \int_0^x (t - x)(0) dt \Longrightarrow y_3 = 0$$

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And repeating this approach, gives

$$y_4 = y_5 = \ldots = 0$$

This implies, the approximation to the solution of example 1 gives;

$$y = \sum_{n=0}^{\infty} y_n = \sin(x) + 0 + 0 + 0 + \ldots$$

Therefore,

$$y(x) = \sin(x)$$

This is the exact solution of Example 1.

CASE 2:

$$y(x) = \sum_{n=0}^{N} a_n x^n \ and \ y(t) = \sum_{n=0}^{N} a_n t^n, \qquad n = 0(1)N \qquad (16)$$

Substituting (16) into (14) gives;

$$\sum_{n=0}^{N} a_n x^n = x + \int_0^x (t-x) \sum_{n=0}^{N} a_n t^n \, dt \qquad (17)$$

Integrating, expanding and simplifying equation (17) gives;

$$\sum_{n=0}^{N} a_n x^n = x - \sum_{n=0}^{N} \frac{a_n x^{n+2}}{(n+1)(n+2)} \qquad (18)$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = x - \frac{a_0 x^2}{2} - \frac{a_1 x^3}{6} - \frac{a_2 x^4}{12} - \frac{a_3 x^5}{20} - \frac{a_4 x^6}{30} - \frac{a_5 x^7}{42} - \ldots \qquad (19)$$

Comparing coefficient of like powers of x gives;

$$a_0 = 0, a_1 = 1, a_2 = 0, a_3 = -\frac{1}{3!}, a_4 = 0, a_5 = \frac{2}{5!}, a_6 = 0, a_7 = -\frac{2}{7!} \ etc$$

Substituting the value of the $a_i's$ in the approximate solution gives;

$$y(x) = x - \frac{x^3}{3!} + \frac{2x^5}{5!} - \frac{2x^7}{7!} + \ldots$$

Table 1: Table of results for Example 1 using Series Solution Method (SSM)

x	Exact	Approximate(SSM)	Error
0	0	0	0
0.1	0.09983341664	0.09983349996	-0.00000008332
0.2	0.1986693309	0.1986719949	−0.0000026640
0.3	0.2955202066	0.2955404132	−0.0000202066
0.4	0.3894183415	0.3895033498	−0.0000850083
0.5	0.4794255333	0.4796843998	−0.0002588665
0.6	0.5646424457	0.5652848914	−0.0006424457

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EXAMPLE 2:

Considering Volterra integral equation

$$y(x) - \int_0^x (x-t)y(t)dt = x + 1 \tag{20}$$

Exact solution $y(x) = e^x$

Source: Shoukralla and Ahmed (2020).

SOLUTION 2:

CASE 1:

Applying the homotopy on equation (12) gives;

$$y(x) = (x+1) + \int_0^x (1-p)(x-t)e^x + p(x-t)y(t)dt \tag{21}$$

Substituting equation (5) into equation (21) and collecting like powers of p gives

$$p^0: y_0 = (x+1) + \int_0^x (x-t)e^t dt \Rightarrow y_0 = e^x$$

$$p^1: y_1 = \int_0^x (-(x-t)e^t + (x-t)e^t)dt \Rightarrow y_1 = 0$$

$$p^2: y_2 = \int_0^x (x-t)(0)dt \Rightarrow y_2 = 0$$

$$p^3: y_3 = \int_0^x (x-t)(0)dt \Rightarrow y_3 = 0$$

And repeating this approach, gives

$$y_4 = y_5 = \dots = 0$$

This implies, the approximation to the solution of example 2 gives;

$$y = \sum_{n=0}^{\infty} y_n = e^x + 0 + 0 + 0 + \dots$$

Therefore,

$$y(x) = e^x$$

This is the exact solution of Example 2.

CASE 2:

Substituting equation (17) into equation (20) gives,

$$\sum_{n=0}^{N} a_n x^n = (x+1) + \int_0^x (x-t)\sum_{n=0}^{N} a_n t^n \, dt \qquad (22)$$

Integrating, expanding and simplifying equation (22) gives;

$$\sum_{n=0}^{N} a_n x^n = (x+1) + \sum_{n=0}^{N} \frac{a_n x^{n+2}}{(n+1)(n+2)} \qquad (23)$$

Expanding equation (23) gives;

$$a_0 + a_1 x + a_2 x^2 ++ a_3 x^3 + \dots$$
$$= (x+1) + \frac{a_0 x^2}{2} + \frac{a_1 x^3}{6} + \frac{a_2 x^4}{12} + \frac{a_3 x^5}{20} + \frac{a_4 x^6}{30} + \frac{a_5 x^7}{42} + \dots \qquad (24)$$

Comparing coefficient of like powers of x gives;

$$a_0 = 1, a_1 = 1, a_2 = \frac{1}{2!}, a_3 =- \frac{1}{3!}, a_4 = \frac{1}{4!}, a_5 = \frac{1}{5!}, a_6 = \frac{1}{6!}, a_7 =- \frac{1}{7!} \; etc$$

Substituting the value of the $a_i's$ in the approximate solution gives;

$$y(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \dots = e^x$$

This gives the exact solution.

EXAMPLE 3:

Considering Volterra integral equation

$$y(x) = 1 + x + \frac{1}{3!}x^3 - \int_0^x (x-t)y(t)dt \qquad (25)$$

Exact solution $y(x) = \mathrm{x} + \cos(\mathrm{x})$

Source: Wazwaz (2011).

SOLUTION 3:

CASE 1:

Applying the homotopy on equation (25) gives;

$$y(x) = 1 + x + \frac{1}{3!}x^3 - \int_0^x (1-p)(x-t)(t+\cos(t)) + p(x-t)y(t)dt \qquad (26)$$

Substituting equation (5) into equation (26) and collecting like powers of p gives

$$p^0: y_0 = \left(1 + x + \frac{1}{3!}x^3\right) - \int_0^x (x-t)(t+\cos(t))dt \Rightarrow y_0 = x + \cos(x)$$

$$p^1: y_1 =- \int_0^x (-(x-t)(t+\cos(t)) + (x-t)(t+\cos(t)))dt \Rightarrow y_1 = 0$$

$$p^2: y_2 =- \int_0^x (x-t)(0)dt \Rightarrow y_2 = 0$$

$$p^3: y_3 =- \int_0^x (x-t)(0)dt \Rightarrow y_3 = 0$$

And repeating this approach, gives

$$y_4 = y_5 = \dots = 0$$

This implies, the approximation to the solution of example 3 gives;

$$y = \sum_{n=0}^{\infty} y_n = (x + \cos(x)) + 0 + 0 + 0 + \dots$$

Therefore,

$$y(x) = (x + \cos(x))$$

This is the exact solution of Example 3.

CASE 2:

Substituting equation (17) into equation (25) gives,

$$\sum_{n=0}^{N} a_n x^n = 1 + x + \frac{1}{3!}x^3 - \int_0^x (x-t) \sum_{n=0}^{N} a_n t^n \, dt \quad (27)$$

Integrating, expanding and simplifying equation (22) gives;

$$\sum_{n=0}^{N} a_n x^n = 1 + x + \frac{1}{3!}x^3 - \sum_{n=0}^{N} \frac{a_n x^{n+2}}{(n+1)(n+2)} \quad (23)$$

Expanding equation (23) gives;

$$a_0 + a_1 x + a_2 x^2 ++ a_3 x^3 + \dots = 1 + x + \frac{1}{3!}x^3 - \left(\frac{a_0 x^2}{2} + \frac{a_1 x^3}{6} + \frac{a_2 x^4}{12} + \frac{a_3 x^5}{20} + \frac{a_4 x^6}{30} + \frac{a_5 x^7}{42} + \dots\right) \quad (24)$$

Comparing coefficient of like powers of x gives;

$$a_0 = 1, a_1 = 1, a_2 =- \frac{1}{2!}, a_3 = 0, a_4 = \frac{1}{4!}, a_5 = 0, a_6 =- \frac{1}{6!}, a_7 = 0 \; etc$$

Substituting the value of the a_i's in the approximate solution gives;

$$y(x) = x + 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$y(x) = x + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) = x + cosx$$

This gives the exact solution.

RESULTS AND DISCUSSION

In this section numerical findings from the solved examples using the proposed numerical approach were discussed. The results obtained for Example 1, Example 2 and Example 3 using Homotopy Perturbation Method gives the exact solution, hence the reason it is not in tabular or graphical form while the results obtained for Example 2 and Example 3 using Series Solution method gives the exact solution too, hence the reason it is not in tabular or graphical form. The result of example 1 using series solution method as seen in Table 1 converge faster to the exact solution. This shows that the Homotopy Perturbation Method performs better and it is more efficient than the Series Solution Method.

The numerical result gives an exact solution, and this confirmed that our method performed better than the method proposed by (Zarnan, 2018), (Shoukralla and Ahmed, 2020) and (Ajileye *et al*., 2024). The Homotopy Perturbation Method and Series Solution Method give accurate solutions to the integral equations but the Homotopy Perturbation Method converges faster than the Series Solution Method while the Series Solution Method is easier and more flexible to implement.

CONCLUSION

In this work, we used homotopy perturbation method and series solution method to solve Volterra integral equations. These methods are simple, reliable, and computationally effective. Maple 18 software is used for all computations in this work. The accuracy of the method is demonstrated by considering some examples.

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