Bima Journal of Science and Technology Vol. 2 No. 1 January, 2018 ISSN 25366041

A STUDY ON THE COMPACTNESS OF SETS

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Abstract:

The general properties of compact sets are looked at in the study such as compactness implies closedness, compactness implies bounded. In order to achieve these we briefly looked at metric spaces, normed vector spaces from which we considered metric that is being induced by a norm. Considered also is the convergence property of some sequences chiefly Cauchy sequence on which the completeness concept of a set consequently continuity and boundedness properties are established and property of a finite dimensional compact sets. Everything looked at is concise.

Keywords:
Normed vector space; compact; convergence sequence; Cauchy sequence; closed; bounded

Introduction

Unlike in other fields, before a set can be referred to as a space in mathematics it must satisfy some criteria (properties on which it should be defined called its axioms), hence a space is a set with structures. Metric space is one of the fundamentals of other space as other space inherit properties from it forming a hierarchy, for instance all normed vector spaces are also metric spaces, because the normed vector induces a metric on the normed vector space such that:

$d(x, y) = \|y - x\|$ for $x, y \in X$

In the real line $\mathbb{R}$ the distance between two points x, y is the absolute value of their difference defined $d(x, y) = |x - y|$ while for vectors x, y the distance is the norm of their difference defined $d(x, y) = \| x - y \|$. The minimal property attained by distance is defined by a metric function. In addition, one has little or no control over unbounded set and absolutely nothing to say on space that is discontinuous everywhere around it neighbourhood. Thus, a compact set or space as the case may be gives a well behave set or space as one could define some standard properties on such a space and this is what we seek to verify. In this section, we introduce some spaces which are used in the study.

Normed Vector Space

A normed vector space $(X, \| . \|)$ is a vector space X equipped with a norm $\| . \|$ i.e a function $\| . \| : X \longrightarrow \mathbb{R}^+ := [0, \infty)$; $x \mapsto$

$\| x\|$ such that for all $x, y \in X$ and $\lambda \in K$ have the following axiomatic definitions

i. $\| x\| \geq 0$ $(nonnegative)$
ii. $\| \lambda x\| = |\lambda| \| x\|$ $(positive\ homogeneity)$
iii. $\|x + y\| \leq \| x\| + \| y\|$ $(subadditivity)$
iv. $\| x\| = 0$ implies $x = 0$ $(positive\ definiteness)$

where X is a real or complex vector space and K a field of real or complex numbers.

Closedness of Sets

A sequence (x_n) of a subset S of a metric space such that $x_n \longrightarrow x$ for each x, and $x_n \neq x$, then x is called a limit point of S.
If S contains all its limit points then it is said to be closed.

Bounded of Sets

A metric space (X, d) is called bounded if there exists a number r such that

$$d(x, y) \leq r, \qquad \forall \quad x, y \in X.$$

Proposition 1.1

Every normed vector space X is a metric space, relative to the natural metric d defined by $d(x, y) = \| y - x\|$ for $x, y \in X$. Furthermore for any x, y, z in X and for all $\lambda \in K$, we have $\|x\| = d(0, x)$ as well as

i. $d(x + z,\ y + z) = d(x, y)$ $Translation\ invariance$
ii. $d(\lambda x, \lambda y) = |\lambda|\, d(x, y)$ $positive\ homogeinity$

Proof

It can be easily verified that the axioms for a metric hold for e.g. $d(x, z) \leq d(x, y) + d(y, z)$ follows immediately by writing $(z - x) = (y - x) + (z - y)$ so that

$$\| x - z \| \leq \|(y - x) + (z - y) \| \leq \| y - x \| + \| z - y \|$$

i. $d(\, x + z,\ y + z\,) = \|(y + z) - (x + z)\| = \|y - x\| = \|x - y\| = d(x, y)$

ii. $d(\lambda x,\ \lambda y) = \|\lambda x - \lambda y\| = \| \lambda(x - y)\| = |\lambda|\|x - y\| = |\lambda|\, d(x, y)$

Every normed vector space is also a metric space if a metric is define via a norm as follows

$$d(x,y) := \| x - y \|$$

In particular this implies $\| x \| = d(x,0) \geq 0$ for all $x \in X$. However, not every metric stems from a norm, for example the discrete metric on R is not induced by a norm.

Result

Compactness can be described invariantly in many ways but in these subject we define it in two ways

Compact space in terms of sequence

A subset Y of a metric space X is compact if every sequence of points x_n in Y, there is a subsequence x_{n_i} which converges to a point x in Y.

Compact space in terms property of closed and bounded property

Theorem 2.1

Compact set are closed and bounded.

Proof

First to show that compact sets are closed, we must show that it contains all its limit points, Let X be a compact set then any sequence of points x_n in X, has a subsequence x_{n_i}which converges to a point in X, let that point be a. Let x be a limit point of X, then X contains a sequence converging to x and all subsequence converges to x. Since all sequence and subsequence of X converges to x, then $a = x$. Hence, X is compact and contains all its limit points.

Second to show that compact set are bounded, a set is bounded if for any sequence (x_n) there is an integer m, such that $\|x_n\| < m$, if X is not bounded then $\|x_n\| > m$ implying a divergence contradicting definition of compactness.

Closed and bounded sets does not necessarily imply that the sets are compacts e.g.;

1. The closed unit ball $\| x \| \leq 1$ in $L_2[0,1]$ is bounded yet not compact since no subsequence of it can converge (Cauchy convergence criterion not satisfied).
2. The standard basis vector $e_1 = (1,0,0,\dots), e_2 = (0,1,0,0,\dots)$ etc have no convergent subsequence, and also
3. The real number $x \in \mathbb{R}, 0 \leq x \leq 1$ with the discrete metric $d(x,y) = 1$ for $x \neq y, d(x,x) = 0$.
4. The set of rational numbers Q.

Even though compact sets are closed and bounded, not all closed and bounded sets are compact.

In general, closed and bounded set are not compact, but in the Euclidean space R^n they are.

$compact \Rightarrow closed + bounded \qquad closed + bounded \nRightarrow compact$

Discussion

We were able to define compact space based on sequence also on closed and boundedness. Hence we can define a normed space to be compact based on sequence as;

Compact Normed Vector Space

Let N be a normed vector space and M a subset of N, then M is compact if every finite sequence of elements of M has a subsequence which converges to an element of M.

We check to see if normed spaces are closed and bounded.

In definition of boundedness we say that a metric space (X, d) is called bounded if there exists a number r such that $d(x, y) \leq r$, $\forall \; x, y \in X$.

It is also a known fact proven from Proposition (1.1) that every normed vector space is a metric space define as $d(x, y) = \| y - x \| \; for \; all \; x, y \in X$.

Since $d(x, y) \leq r$ and $d(x, y) = \| y - x \|$ this implies $d(x, y) = \| y - x \| \leq r$ i.e. $\| y - x \| \leq r$.

Bounded Normed Space

A normed vector space N is said to be bounded if there exist a number r such that

$$\| y - x \| \leq r, \quad \forall \; x, y \in N.$$

Closed Normed Space

A normed vector space N is said to be closed if it contains all its limit points.

Since normed vector space can be closed and bounded, therefore normed vector spaces are compact.

Convergence Sequence in Normed Spaces

A sequence (x_n) of a normed space N, is said to converge if given any $\varepsilon > 0$, there is a number M such that $\| x_n - x \| < \varepsilon$ for all $n > M$. We say $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

Theorem 3.2 A convergent sequence in R^n can have at most one limit.

Proof

A sequence x_n has a limit x if for any $\varepsilon > 0$ there is an $N(\varepsilon)$ such that $|x_n - x| < \varepsilon$ for all $n \geq N(\varepsilon)$ written as $x_n \rightarrow x$.

Suppose that by contradiction the sequence x_n has two limit x and x' such that

$$|x_n - x| < \frac{\varepsilon}{2} \text{ for all } n \geq N(\varepsilon)$$

$$|x_n - x'| < \frac{\varepsilon}{2} \quad \text{for all } n \geq N(\varepsilon)$$

By triangle inequality we have;

$$|x - x'| \leq |x - x_n| + |x_n - x'|$$

$$\leq |x_n - x| + |x_n - x'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

Since it holds for any $\varepsilon > 0$ then we have that

$$|x - x'| = 0$$

Hence $x = x'$

Theorem 3.3 Bolzano-Weierstrass

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Proof

We note that $\mathbb{R}^n$ is finite dimensional Banach space and compact. Thus, from definition under section (2.1) the result follows immediately since every sequence in $\mathbb{R}^n$ converges to element in it and since it is finite, it is bounded. The convergent subsequence comes from the fact $\mathbb{R}^n$ is compact.

Cauchy sequence of a normed vector space

A sequence (x_n) of a normed vector space N, is said to be Cauchy sequence of a normed vector if given any $\varepsilon > 0$, there is a number M such that $\| x_n - x_m \| < \varepsilon$ for all $n, m > M$.

Theorem 3.4 Every convergent sequence in a normed space is a Cauchy sequence.

Proof

Let $\| x_n, x \|$ be a normed space and (x_n) be a sequence converging to a point x in X

i.e., $x_n \to x$, for any $\epsilon > 0$ $\exists\ n_0$ such that $\|x_n, x\| < \frac{\epsilon}{2}$ for $n \geq n_0$.

Let $n, m \geq n_0$. Then $\|x_n, x_m\| \leq \|x_n, x\| + \|x, x_m\| = \|x_n, x\| + \|x_m, x\| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\therefore (x_n)$ is Cauchy.

We note that although all convergent sequence are Cauchy yet not all Cauchy sequence are convergent.

For instance let $X = (0,1)$ be an open interval and $Y = [0,1]$ be a close interval with the usual norm defined on them, let $x_n = \frac{1}{n}$ be a sequence, this sequence is Cauchy converging to a point 0 which is in Y but not in X.

$0 \in [0,1]$ but $0 \notin (0,1)$

i.e., It converges to a point in Y and a point not in X. Hence not all Cauchy sequence are convergent.

Careful observation on why this Cauchy sequence is not convergent is due to the domain of interval on which it is defined. So we postulate the next theorem;

Theorem 3.5 Every Cauchy sequence in a normed space in $\mathbb{R}^n$ are convergent if and only if the domain on which the interval is defined is compact.

Proof

($\Longrightarrow$) Let (x_n) be a Cauchy sequence which is convergent then we show that it is compact i.e. closed and bounded. Since it is a convergent sequence, it converge to one point say x which is it limit point and from Theorem (3.2) (A convergent sequence in R^n can have at most one limit), therefore it contains all its limit points, hence closed.

Since it is convergent, it is bounded i.e. for an integer m, $\|x_n\| < m$ else $\|x_n\| > m$ which is divergent and a contradiction to being convergent sequence. Hence it is compact.

($\Longleftarrow$) We assume that the sequence (x_n) is compact and show that it is convergent and subsequently Cauchy. Since it is compact, it is bounded. By Bolzano-Weierstrass theorem every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence and from theorem 3 every convergent sequence in a normed space is a Cauchy sequence.

Conclusion

In general, Compactness is not hereditary because even though (0,1) is a subset of [0,1], is not a compact subset of [0,1] on the other hand it is closed hereditary as any closed subset of a compact set is compact, the generality of normed space are preserved when the domain on which it is defined is compact.

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