Bima Journal of Science and Technology, Vol. 8(3) Sept, 2024 ISSN: 2536-6041

DOI: 10.56892/bima.v8i3A.811

On Multi Points of Multisets

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ABSTRACT

This paper presents a review of operations on collections of multi-points of multisets involving the addition, union, intersection, difference, symmetric difference, complement, arithmetic multiplication, raising to arithmetic power and scalar multiplication. Furthermore, some algebraic properties of these operations such as commutativity, idempotency, associativity, distributivity and De Morgan's laws are studied. The concept of a root set multi-point collection of a multipoint collection is examined under union and intersection of such collections.

Keywords: Multiset, Multi-Point, Multi Real Point, De Morgan's Law, Root Set

INTRODUCTION

Having objects with common attribute in science and mathematics is a natural phenomenon and mostly, researchers are interested only in the number of such objects (elements) sharing particular property Syropoulos (2003). For instance molecules of a particular compound, roots of polynomials, sampling data, etc. The concept of multisets (msets for short) is a generalization of a classical set. In Cantor's sets objects are not allowed to repeat, if the repetition is allowed then a mathematical structure known as msets emerged. An msets is an unordered collection of objects in which, objects are allowed to repeat. The occurrence of individual object in the mset is called its element; and each indistinguishable element might be finite number in the mset. The number of occurrences of an individual element in a mset is called the multiplicity of the element which is finite in most of the applications and contribute to the cardinality of the mset as discussed by Blizard (1989) and Ibrahim et al. (2011).

The notion of multi points (mpoints, for short) and multi real points (mrpoints, for short) was first introduced by Das and Roy (2021) where the definition of mpoints, mrpoints and some basic properties were presented. However, rich as it is, this concept approach was limited to the study of multi metric spaces and applications. This work intends to review Das and Roy (2021) and incorporate some notions. We present some preliminaries and notations on msets and collections of mpoints in section two. In section three, we present some operations on collections of mpoints such as addition, union, intersection, difference, complement, symmetric difference, arithmetic multiplication, raising to arithmetic power, and scalar multiplication and examine the basic algebraic properties of these operations such as commutativity, idempotency, associativity, distributivity and absorption law. We examine De Morgan's laws as applied to the complementation of union and intersection of collections of mpoints as well as the root set of collection of mpoint on union and intersection.

PRELIMINARIES

msets

Definition 2.1.1(mset) Debnath and Debnath (2019)

A collection of elements which are allowed to repeat is called a mset. Formally if $\mathcal{S}$ is a set of elements, an mset A drawn from the set $\mathcal{S}$ is represented by a function $C_A: \mathcal{S} \to \mathbb{N}$, where $\mathbb{N}$ represents the set of non-negative integers. For each $x \in \mathcal{S}$, $C_A(x)$ is the characteristic value of x in A and indicates the number of occurrences of the element x in A. A mset A is a set if

$$C_A(x) \in \{0,1\}\ \forall x \in \mathcal{S}.$$

The symbolic representation $x \in^k A$ means x belongs to A exactly k times.

Here, we presents mset M drawn from the set $X = \{x_1, x_2, x_3, \ldots x_n\}$ as

$M = \{x_1, x_2, x_3, \ldots x_n\}_{m_1, m_2, m_3, \ldots, m_n}$, or $M = \{m_1/x_1, m_2/x_2, \ldots, m_n/x_n\}$ where m_i are the multiplicities of the elements x_i, $i = 1,2, \ldots, n$

Example 2.1.2

Let $X = \{a, b, c, d, e\}$. Then $M = \{a, b, d, e\}_{2,4,5,1} = \{2/a, 4/b, 5/d, 1/e\}$ is a mset drawn from X.

Definition 2.1.3 (Cardinality of a mset) Debnath and Debnath (2019).

The cardinality of a mset A drawn from a set $\mathcal{S}$ denoted by card A is defined by

card $A = \sum_{x \in \mathcal{S}} C_A(x)$. It is also denoted by $|A|$. A mset A over the set $\mathcal{S}$ is said to be finite if $|A| < \infty$. We denote the class of all finite msets over $\mathcal{S}$ by $\mathfrak{M}(\mathcal{S})$.

Definition 2.1.4 (root set) Debnath and Debnath (2019)

Let $A \in \mathfrak{M}(\mathcal{S})$. Then the root set or support set of A denoted A^* is given by the expression

$A^* = \{x \in \mathcal{S} | C_A(x) > 0\}$. For instance, if $A = [a, a, b, b, b, c, c, c, c]$, then $A^* = \{a, b, c\}$. For every x such that $C_A(x) = 0$ implies $x \notin A$.

Note that $x \in A \leftrightarrow C_A(x) > 0$ and $x \in A \leftrightarrow x \in A^*$

Definition 2.1.5 (empty mset) Petrovsky (2004)

Let $A \in \mathfrak{M}(\mathcal{S})$. Then A is said to be empty mset if $C_A(x) = 0\ \forall x \in \mathcal{S}$. We denote the empty mset by $\emptyset$ and $C_\emptyset(x) = 0\ \forall x \in \mathcal{S}$

Definition 2.1.6 (equality of msets) Tella (2020)

Two msets $A, B \in \mathfrak{M}(\mathcal{S})$ are equal denoted by $A \doteq B$ if $C_A(x) = C_B(x)\ \forall x \in \mathcal{S}$.

Definition 2.1.7 (submset) Tella (2020)

Let $A, B \in \mathfrak{M}(\mathcal{S})$. The mset A is a submultiset (submset for short) of B denoted by $A \sqsubseteq B$ or $B \sqsupseteq A$ if $C_A(x) \leq C_B(x)\ \forall x \in \mathcal{S}$.

If $A \sqsubseteq B$ and $A \neq B$, then A is called the proper submset of B denoted $A \sqsubset B$

Definition 2.1.8 (mset operations) Tella (2020)

i. mset addition

Let $A, B \in \mathfrak{M}(\mathcal{S})$. The mset addition denoted by $A \oplus B$ is given by

$$C_{A \oplus B}(x) = C_A(x) + C_B(x), \forall x \in \mathcal{S}.$$

ii. set difference

Let $A, B \in \mathfrak{M}(\mathcal{S})$.The difference of B from A denoted by $A \ominus B$ is given by

$C_{A\ominus B}(x) = Max\{C_A(x) - C_B(x), 0\}\ \forall x \in \mathcal{S}$.

iii. mset Union

Let $A, B \in \mathfrak{M}(\mathcal{S})$. Then the union of A and B denoted by $A \uplus B$ is defined by

$C_{A\uplus B}(x) = max\{C_A(x), C_B(x)\}\ \forall x \in \mathcal{S}$.

iv. mset Intersection

Let $A, B \in \mathfrak{M}(\mathcal{S})$.Then the intersection of A and B denoted $A \Cap B$ is defined by

$C_{A\Cap B}(x) = min\{C_A(x), C_B(x)\}\forall x \in \mathcal{S}$

v. Symmetric Difference

Let $A, B \in \mathfrak{M}(\mathcal{S})$.Then the symmetric difference of A and B denoted $A\overline{\Delta}B$ is defined by $C_{A\overline{\Delta}B}(x) = |C_A(x) - C_B(x)|$.

Note that $A\overline{\Delta}B = (A \ominus B) \uplus (B \ominus A)$

vi. mset complement

Let $G = \{A_1, A_2, A_3, \ldots, A_n | A_i \in \mathfrak{M}(\mathcal{S})\}$ and $Z = \uplus A_i$. Then the complement for each $A \in G$ denoted by $\overline{A}$ is defined by

$C_{\overline{A}}(x) = C_Z(x) - C_A(x)\forall x \in \mathcal{S}$

Note that for each $A \in G$, we have $A \underline{\sqsubseteq} Z$ and $A \Cap \overline{A} \neq \emptyset$ in general

vii. mset scalar multiplication

$A \in \mathfrak{M}(\mathcal{S})$ and $\propto \in \mathbb{Z}^+$. The scalar multiplication of an mset A denoted $\propto \odot A$

is defined by $C_{\propto\odot A} = \propto \cdot C_A(x), \forall x \in \mathcal{S},\ \ \propto \in \mathbb{Z}^+$

viii. mset arithmetic multiplication

Let $A, B \in \mathfrak{M}(\mathcal{S})$. Then the arithmetic multiplication of A and B denoted by $A \odot B$ is defined by $C_{A\odot B}(x) = C_A(x) \cdot C_B(x)\ \forall x \in \mathcal{S}$

ix. raising to arithmetic power

Let $A \in \mathfrak{M}(\mathcal{S})$. Then the mset A raising to arithmetic power of n denoted by A^n is defined by $C_{A^n}(x) = \left(C_A(x)\right)^n\ \forall x \in \mathcal{S}, n \in \mathbb{Z}^+$

Note that $A^0 = A^*$ (see Gambo and Tella (2022)),

$(A \uplus B)^* = A^* \cup B^*$ and $(A \Cap B)^* = A^* \cap B^*$ (Yager (1986))

x. mset direct product

Let $A, B \in \mathfrak{M}(\mathcal{S})$. Then the direct product of A and B denoted by $A \otimes B$ is defined by $C_{A\otimes B}\big((x, y)\big) = \big\{(a/x, b/y)/a.b\colon\ x \in^a A, y \in^b B\big\}$.

xi. mset raising to the direct power

Let $A \in \mathfrak{M}(\mathcal{S})$. Then the mset A raising to the direct power of n denoted by $(\times A)^n$ is defined by $C_{(\times A)^n}(x_1, x_2, x_3, \ldots, x_n) = \prod_{i=1}^{n} C_A(x_i)$, $x_i \in A$.

MULTI POINTS

Definition 2.2.1 (Multi point) Das and Roy (2021)

Let $M \in \mathfrak{M}(\mathcal{S})$ be a mset over a universal set $\mathcal{S}$. Then a multi point (mpoint, for short) of M is defined by a mapping $P_x^k : \mathcal{S} \rightarrow \mathbb{N}$ such that $P_x^k(x) = k$ where k is unique and $k \leq C_M(\boldsymbol{x})$.

Note that x and k will be referred to as the base and the multiplicity of the mpoint P_x^k respectively. We denote the collection of all mpoints of a mset $M \in \mathfrak{M}(\mathcal{S})$ by M_{pt} and $P_x^k \in M_{pt}$ if $k > 0$ otherwise $P_x^k \notin M_{pt}$ and that M_{pt} is empty denoted $\emptyset_{pt}$ and $P_x^k \in \emptyset_{pt} \rightarrow k = 0 \; \forall x \in \mathcal{S}$

Definition 2.2.2 Das and Roy (2021)

The mset generated by a collection B_{pt} of mpoints is denoted by $MS(B_{pt})$ and is defined by $C_{MS(B_{pt})}(x) = Sup\{k : P_x^k \in B_{pt}\}$

A mset can be generated from the collection of its mpoints. If M_{pt} denotes the collection of all mpoints of $M \in \mathfrak{M}(\mathcal{S})$, then obviously $C_M(x) = Sup\{k : P_x^k \in M_{pt}\}$ since $C_M(x)$ is finite and hence $M = MS(M_{pt})$.

Note that for all set $\mathcal{S}$, $C_{\mathcal{S}}(x) = 1 \forall x \in \mathcal{S}$. Thus $\mathcal{S}_{pt} = \{P_x^1 | x \in \mathcal{S}\}$ and for any $M \in \mathfrak{M}(\mathcal{S})$,

M_{pt} is called finite collection of mpoints of the mset M. Also, $MS(\mathcal{S}_{pt}) = \mathcal{S}$

We shall denote the class of all finite collections of mpoints over $\mathcal{S}_{pt}$ by $\mathfrak{M}(\mathcal{S}_{pt})$

Note that $M \in \mathfrak{M}(\mathcal{S}) \leftrightarrow M_{pt} \in \mathfrak{M}(\mathcal{S}_{pt})$.

Definition 2.2.3

Let $A_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. The root set of A_{pt} denoted A_{pt}^* is defined:

$A_{pt}^* = \{P_x^1 \in \mathcal{S}_{pt} | x \in A^* \wedge P_x^k \in A_{pt}\} \forall x \in \mathcal{S}$.

Note that for any P_x^k and P_y^l we have $P_x^k = P_y^l \leftrightarrow x = y \wedge k = l$,

$P_x^1 \in A_{pt}^* \leftrightarrow x \in A^* \wedge P_x^k \in A_{pt}$ and $A_{pt}^* \subseteq \mathcal{S}_{\boldsymbol{pt}}$

Definition 2.2.4 Das and Roy (2021)

i. Let $C_{pt}, D_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. The elementary union between C_{pt} and D_{pt} denoted by $C_{pt} \sqcup D_{pt}$ is defined: $C_{pt} \sqcup D_{pt} = \{P_x^k : P_x^l \in C_{pt}, P_x^m \in D_{pt} \; and \; k = \max\{l, m\}\}$
ii. Let $C_{pt}, D_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. The elementary intersection between C_{pt} and D_{pt} denoted by $C_{pt} \sqcap D_{pt}$ is defined: $C_{pt} \sqcap D_{pt} = \{P_x^k : P_x^l \in C_{pt}, P_x^m \in D_{pt} \; and \; k = \min\{l, m\}\}$
iii. Let $C_{pt}, D_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. C_{pt} is said to be an elementary subset of D_{pt}, denoted by $C_{pt} \sqsubseteq D_{pt} \leftrightarrow P_x^l \in C_{pt} \Longrightarrow \exists! \, m \geq l \; such \; that \; P_x^m \in D_{pt}$

Definition 2.2.5

Let $C_{pt}, D_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. Then C_{pt} is said to be equal to D_{pt} denoted $C_{pt} = D_{pt}$ and defined as:
$C_{pt} = D_{pt} \rightarrow (P_x^k \in C_{pt} \leftrightarrow P_x^k \in D_{pt})\ \forall x \in \mathcal{S}$

Proposition 2.2.6

Let $C_{pt}, D_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. Then $C_{pt} \sqsubseteq D_{pt} \wedge D_{pt} \sqsubseteq C_{pt}$ iff $C_{pt} = D_{pt}$

Proof

If Let $C_{pt}, D_{pt} \in \mathfrak{M}(\mathcal{S}_{pt})$, such that $C_{pt} \sqsubseteq D_{pt}$ and $D_{pt} \sqsubseteq C_{pt}$

But $C_{pt} \sqsubseteq D_{pt} \ \leftrightarrow\ P_x^l \in C_{pt} \Longrightarrow \exists m(m \geq l\ and\ P_x^m \in D_{pt})$ (1)

also $D_{pt} \sqsubseteq C_{pt} \Longrightarrow l \geq m$ since $P_x^l \in C_{pt}$ and $P_x^m \in D_{pt}$ and l is unique (2)

Now from (1) and (2) above, we have $m = l$ (3)

Thus from (1), (2) and (3) above, we have

$(C_{pt} \sqsubseteq D_{pt} \wedge D_{pt} \sqsubseteq C_{pt}) \rightarrow (P_x^l \in C_{pt} \leftrightarrow P_x^l \in D_{pt})$ (4)

In particular, $C_{pt} = D_{pt}$ (from definition 2.2.5)

Definition 2.2.7 (Addition of mpoint)

Let $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. The mpoint addition of A_{pt}, B_{pt} denoted by $A_{pt} \oplus B_{pt}$ is define by

$A_{pt} \oplus B_{pt} = \{P_x{}^k | P_x^m \in A_{pt}, P_x^l \in B_{pt} \wedge k = m + l\}$

Definition 2.2.8 (Difference of mpoint)

Let $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. The mpoint difference of A_{pt}, B_{pt} denoted by $A_{pt} \ominus B_{pt}$ is define by

$A_{pt} \ominus B_{pt} = \{P_x^k | P_x^m \in A_{pt}, P_x^l \in B_{pt} \wedge\ k = max\{m - l, 0\}\}$

Note that if $B_{pt} \sqsubseteq A_{pt}$ then $A_{pt} \ominus B_{pt} = \{P_x^k | P_x^m \in A_{pt}, P_x^l \in B_{pt} \wedge\ k = m - l\}$

Definition 2.2.9 (symmetric difference of mpoint)

Let $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. The mpoint symmetric difference of A_{pt}, B_{pt} denoted by $A_{pt}\overline{\Delta}B_{pt}$ is define by

$A_{pt}\overline{\Delta}B_{pt} = \{P_x^k | P_x^m \in A_{pt}, P_x^l \in B_{pt} \wedge k = |m - l|\}$

Definition 2.2.10 (complement of mpoint)

Let $A_{1pt}, A_{2pt}, \ldots, A_{npt}: A_{ipt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$, if $Z_{pt} = \sqcup A_{ipt}$. The mpoint complement of each A_{ipt} denoted by $\overline{A_{ipt}}$ is defined by

$\overline{A_{ipt}} = Z_{pt} \ominus A_{ipt}$

Note that $A_{ipt} \sqsubseteq Z_{pt}$

DOI: 10.56892/bima.v8i3A.811

Definition 2.2.11 (arithmetic multiplication of mpoint)

Let $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. The arithmetic multiplication of mpoints A_{pt}, B_{pt} denoted by $A_{pt} \odot B_{pt}$ is define by

$A_{pt} \odot B_{pt} = \{P_x^k | P_x^m \in A_{pt}, P_x^l \in B_{pt} \ \wedge k = m.l\}$

Definition 2.2.12 (raising to arithmetic power)

Let $A_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. The mpoint A_{pt} raising to arithmetic power n denoted by A_{pt}^n is define by: $A_{pt}^n = \{P_x^k | P_x^l \in A_{pt} \wedge k = l^n\}$

Definition 2.2.13 (scalar multiplication)

Let $A_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$ and $\propto \in \mathbb{Z}^+$. The mpoint scalar multiplication of A_{pt} denoted by $\propto A_{pt}$ is define by: $\propto A_{pt} = \{P_x^k | P_x^l \in A_{pt}, \propto \ \in \mathbb{Z}^+ \wedge k = \propto . l\}$

Definition 2.2.14 (cross product)

Let $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. The direct product of mpoints A_{pt}, B_{pt} denoted by $A_{pt} \otimes B_{pt}$ is defined by $A_{pt} \otimes B_{pt} = \{P_{(x,y)}^k: \ , k = r.l \text{ where } P_x^r \in A_{pt}, P_y^l \in B_{pt}\}$

Proposition 2.2.15. Let $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. Then

i. $MS(A_{pt} \sqcup B_{pt}) = A \uplus B$
ii. $MS(A_{pt} \sqcap B_{pt}) = A \Cap B$
iii. $MS(A_{pt} \oplus B_{pt}) = A \oplus B$
iv. $MS(A_{pt} \ominus B_{pt}) = A \ominus B$
v. $MS(A_{pt} \odot B_{pt}) = A \odot B$
vi. $MS(A_{pt} \overline{\Delta} B_{pt}) = A \overline{\Delta} B$

Proof:

(i). Now $C_{MS(A_{pt} \sqcup B_{pt})}(x) = sup\{k: P_x^k \in A_{pt} \sqcup B_{pt}\}$ (by definition)

But $sup\{k: P_x^k \in A_{pt} \sqcup B_{pt}\} = C_{A \uplus B}(x)$ (by definition)

Thus, $C_{MS(A_{pt} \sqcup B_{pt})}(x) = C_{A \uplus B}(x)$

In particular, $MS(A_{pt} \sqcup B_{pt}) = A \uplus B$

(ii). $C_{MS(A_{pt} \sqcap B_{pt})}(x) = sup\{k: P_x^k \in A_{pt} \sqcap B_{pt}\}$ (by definition)

But $sup\{k: P_x^k \in A_{pt} \sqcap B_{pt}\} = C_{A \Cap B}(x)$ (by definition)

Thus, $C_{MS(A_{pt} \sqcap B_{pt})}(x) = C_{A \Cap B}(x)$

In particular, $MS(A_{pt} \sqcap B_{pt}) = A \Cap B$

(iii-vi) follow similar proofs

Proposition 2.2.16 . Let $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. Then

(i). $A \underline{\subseteq} B \to A_{pt} \subseteq B_{pt}$ and

(ii). $A_{pt} \sqsubseteq B_{pt} \to A^*_{pt} \subseteq B^*_{pt}$

Proof:

(i). Let $A \underline{\subseteq} B$ and $P_x^k \in A_{pt}$.

Clearly, $k \leq C_A(x) \leq C_B(x)$ (by definition and hypothesis)

and $k \leq C_B(x)$. Thus, $P_x^k \in B_{pt}$.In particular, $A_{pt} \subseteq B_{pt}$.

(ii). let $P_x^1 \in A^*_{pt}$ and $P_x^k \in A_{pt}$. We have $x \in A^*$ and $C_A(x) > 0$ (By definition)

Clearly, $P_x^l \in B_{pt}$ where $k \leq l$(by definition and hypothesis)

Thus, $P_x^1 \in B^*_{pt}$ and $A_{pt} \sqsubseteq B_{pt} \to A^*_{pt} \subseteq B^*_{pt}$.

PROPERTIES OF ALGEBRAIC OPERATIONS ON $\mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$

With reference to the operations defined on the space $\mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$, we examine the basic algebraic properties of these operations such as commutativity, idempotency, associativity and distributivity and De Morgan's laws as applied to the complementation of union and intersection.

Proposition 3.1
Let $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. $A_{pt} \sqcup B_{pt} \sqsubseteq (A \uplus B)_{pt}$
Proof.
Let $P_x^k \in A_{pt} \sqcup B_{pt}$ and $P_x^m \in A_{pt}, P_x^L \in B_{pt}$ such that $k = max\{m, l\}$.
But $m \leq C_A(x)$ and $l \leq C_B(x)$ (by definition)
Thus, $k = max\{m, l\} \leq max\{C_A(x), C_B(x)\} = C_{A \uplus B}(x)$
and $P_x^k \in (A \uplus B)_{pt}$.
Thus, $A_{pt} \sqcup B_{pt} \sqsubseteq (A \uplus B)_{pt}$
However, the converse $(A \uplus B)_{pt} \sqsubseteq A_{pt} \sqcup B_{pt}$ is not always true. For example:
Let $A = [x, y]$ and $B = [y, y]$ be two msets, and their union $A \uplus B = [x, y, y]$.
The respective mpoints of A, B and $A \uplus B$ are
$A_{pt} = [P_x^1, P_y^1], B_{pt} = [P_y^1]$ and $(A \uplus B)_{pt} = [P_x^1, P_y^2]$
Now, $A_{pt} \sqcup B_{pt} = [P_x^1, P_y^1]$ and $(A \uplus B)_{pt} \not\sqsubseteq A_{pt} \sqcup B_{pt}$
But $A_{pt} \sqcup B_{pt} = [P_x^1, P_y^1] \sqsubseteq [P_x^1, P_y^2] = (A \uplus B)_{pt}$
Proposition 3.2
Let $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{pt})$. Then $A_{pt} \sqcap B_{pt} \subseteq (A \Cap B)_{pt}$
Proof
Let $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{pt})$.
Let $P_x^k \in A_{pt} \sqcap B_{pt}$, $P_x^m \in A_{pt}$ and $P_x^L \in B_{pt}$ such that $k = min\{m, l\}$
Clearly $P_x^m \in A_{pt}$ implies $m \leq C_A(x)$ and $P_x^L \in B_{pt}$ implies $l \leq C_B(x)$ (by definition)
Thus, $k = min\{m, l\} \leq min\{C_A(x), C_B(x)\} = C_{A \Cap B}(x)$ (by definition)

Bima Journal of Science and Technology, Vol. 8(3) Sept, 2024 ISSN: 2536-6041

DOI: 10.56892/bima.v8i3A.811

Thus, $P_x^k \in (A \cap B)_{pt}$ and $A_{pt} \sqcap B_{pt} \sqsubseteq (A \cap B)_{pt}$

Also here, the converse $(A \cap B)_{pt} \sqsubseteq A_{pt} \sqcap B_{pt}$ is not always true. For example

Let $A = [x, x, y, y, y]$ and $B = [x, x, y, y, z]$ be two msets, and their intersection $A \cap B = [x, x, y, y]$.

The respective mpoints of A, B and $A \cap B$ are

$A_{pt} = [P_x^2, P_y^2], B_{pt} = [P_x^1, P_y^1, P_z^1]$ and $(A \cap B\,)_{pt} = [P_x^1, P_y^2]$

Now, $A_{pt} \sqcap B_{pt} = [P_x^1, P_y^1]$

Hence $(A \cap B)_{pt} \sqsubseteq A_{pt} \sqcap B_{pt}$ is not always true. i.e $(A \cap B)_{pt} \not\sqsubseteq A_{pt} \sqcap B_{pt}$.

However, $A_{pt} \sqcap B_{pt} = [P_x^1, P_y^1] \sqsubseteq [P_x^1, P_y^2] = (A \cap B)_{pt}$

Proposition 3.3. Let $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{pt})$. Then

$A_{pt} \odot B_{pt} \sqsubseteq (A \odot B)_{pt}$

Proof

Let $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{pt})$.

Let $P_x^k \in A_{pt} \odot B_{pt}$ and $P_x^m \in A_{pt}, P_x^l \in B_{pt}$ such that $k = ml$ (by definition)

But $m \leq C_A(x)$ and $l \leq C_B(x)$ (by definition)

Thus, $k = ml \leq C_A(x).C_B(x) = C_{A \odot B}(x)$

In particular, $P_x^k \in (A \odot B)_{pt}$ and $A_{pt} \odot B_{pt} \sqsubseteq (A \odot B)_{pt}$

However, $(A \odot B)_{pt} \sqsubseteq A_{pt} \odot B_{pt}$ is not always true. For example

Let $A = [x, x, y, y, y]$ and $B = [x, x, y, y, z]$ be two msets, and their arithmetic multiplication $A \odot B = [x, x, x, x, y, y, y, y, y, y]$.

The respective mpoints of A, B and $A \odot B$ are

$A_{pt} = [P_x^1, P_y^2], B_{pt} = [P_x^2,, P_y^1, P_z^1]$ and $(A \odot B\,)_{pt} = [P_x^3, P_y^4]$

Now, $A_{pt} \odot B_{pt} = [P_x^2, P_y^2]$

Hence $(A \odot B)_{pt} = [P_x^3, P_y^4] \not\sqsubseteq [P_x^2, P_y^2] = A_{pt} \odot B_{pt}$

However $A_{pt} \odot B_{pt} = [P_x^2, P_y^2] \sqsubseteq [P_x^3, P_y^4] = (A \odot B)_{pt}$

Proposition 3.4 (De Morgan's law of mpoints)

For any $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. Then

i. $[A_{pt} \sqcap B_{pt}]^c = {A_{pt}}^c \sqcup {B_{pt}}^c$

ii. $[A_{pt} \sqcup B_{pt}]^c = {A_{pt}}^c \sqcap {B_{pt}}^c$

Proof

i. Let $P_x^k \in [A_{pt} \sqcap B_{pt}]^c$ and $P_x^l \in Z_{pt}, P_x^m \in A_{pt}, P_x^r \in B_{pt}$ such that $k = l - min\{m, r\}$
   But $k = l - min\{m, r\} = max\{l - m, l - r\}$.
   Thus, $[A_{pt} \sqcap B_{pt}]^c = (Z_{pt} - A_{pt}) \sqcup (Z_{pt} - B_{pt})$
   In particular, $[A_{pt} \sqcap B_{pt}]^c = {A_{pt}}^c \sqcup {B_{pt}}^c$
ii. Let $P_x^k \in [A_{pt} \sqcup B_{pt}]^c$ and $P_x^l \in Z_{pt}, P_x^m \in A_{pt}, P_x^r \in B_{pt}$ such that $k = l - max\{m, r\}$
   But $k = l - max\{m, r\} = min\{l - m, l - r\}$.
   $[A_{pt} \sqcup B_{pt}]^c = (Z_{pt} - A_{pt}) \sqcap (Z_{pt} - B_{pt})$.
   In particular, $[A_{pt} \sqcup B_{pt}]^c = {A_{pt}}^c \sqcap {B_{pt}}^c$

Bima Journal of Science and Technology, Vol. 8(3) Sept, 2024 ISSN: 2536-6041

DOI: 10.56892/bima.v8i3A.811

Proposition 3.5

For any $A_{pt}, B_{pt}, C_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. Then

i. $A_{pt} \sqcup (B_{pt} \sqcup C_{pt}) = (A_{pt} \sqcup B_{pt}) \sqcup C_{pt}$

ii. $A_{pt} \sqcap (B_{pt} \sqcap C_{pt}) = (A_{pt} \sqcap B_{pt}) \sqcap C_{pt}$

iii. $A_{pt} \sqcup (B_{pt} \sqcap C_{pt}) = (A_{pt} \sqcup B_{pt}) \sqcap (A_{pt} \sqcup C_{pt})$

iv. $A_{pt} \sqcap (B_{pt} \sqcup C_{pt}) = (A_{pt} \sqcap B_{pt}) \sqcup (A_{pt} \sqcap C_{pt})$

Proof

i. Let $P_x^k \in A_{pt} \sqcup (B_{pt} \sqcup C_{pt}), P_x^l \in A_{pt}, P_x^m \in B_{pt}, P_x^r \in C_{pt}$ such that $k = max\{l, max\{m, r\}\}$(by definition)
But $k = max\{l, max\{m, r\}\} = max\{max\{l, m\}, r\} \ \forall x$

Thus, $A_{pt} \sqcup (B_{pt} \sqcup C_{pt}) = (A_{pt} \sqcup B_{pt}) \sqcup C_{pt}$ (by definition)

ii. Let $P_x^k \in A_{pt} \sqcap (B_{pt} \sqcap C_{pt}), P_x^l \in A_{pt}, P_x^m \in B_{pt}, P_x^r \in C_{pt}$ such that $k = min\{l, min\{m, r\}\}$(by definition)
But $k = min\{l, min\{m, r\}\} = min\{min\{l, m\}, r\} \ \forall x$

In particular, $A_{pt} \sqcap (B_{pt} \sqcap C_{pt}) = (A_{pt} \sqcap B_{pt}) \sqcap C_{pt}$ (by definition)

iii. Let $P_x^k \in A_{pt} \sqcup (B_{pt} \sqcap C_{pt}), P_x^l \in A_{pt}, P_x^m \in B_{pt}, P_x^r \in C_{pt}$ such that $k = max\{l, min\{m, r\}\}$(by definition)
But $k = max\{l, min\{m, r\}\} = min\{max\{l, m\}, max\{l, r\}\} \ \forall x$

Thus, $A_{pt} \sqcup (B_{pt} \sqcap C_{pt}) = (A_{pt} \sqcup B_{pt}) \sqcap (A_{pt} \sqcup C_{pt})$ (by definition)

iv. Let $P_x^k \in A_{pt} \sqcap (B_{pt} \sqcup C_{pt}), P_x^l \in A_{pt}, P_x^m \in B_{pt}, P_x^r \in C_{pt}$ such that $k = min\{l, max\{m, r\}\}$(by definition)
But $k = min\{l, max\{m, r\}\} = max\{min\{l, m\}, min\{l, r\}\} \ \forall x$

Thus, $A_{pt} \sqcap (B_{pt} \sqcup C_{pt}) = (A_{pt} \sqcap B_{pt}) \sqcup (A_{pt} \sqcap C_{pt})$ (by definition)

Proposition 3.7 (Absorption laws of mpoint)

For any $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. Then

i. $A_{pt} \sqcup (A_{pt} \sqcap B_{pt}) = A_{pt}$

ii. $A_{pt} \sqcap (A_{pt} \sqcup B_{pt}) = A_{pt}$

Proof

i. Let $P_x^k \in A_{pt} \sqcup (A_{pt} \sqcap B_{pt}), P_x^m \in A_{pt}, P_x^r \in B_{pt}$,

such that $k = max\{m, min\{m, r\}\}$ (by definition) (1)

if $min\{m, r\} = r$, then $max\{m, r\} = m$ (from (1)) (2)

if $min\{m, r\} = m$, then $max\{m, min\{m, r\}\} = max\{m, m\} = m$ (3)

Thus, $k = m \ \forall x$(from 1-3)

Bima Journal of Science and Technology, Vol. 8(3) Sept, 2024 ISSN: 2536-6041

DOI: 10.56892/bima.v8i3A.811

In particular, $A_{pt} \sqcup (A_{pt} \sqcap B_{pt}) = A_{pt}$(by definition)

ii. Let $P_x^k \in A_{pt} \sqcap (A_{pt} \sqcup B_{pt}), P_x^m \in A_{pt}, P_x^r \in B_{pt}$,

such that $k = min\{m, max\{m, r\}\}$ (by definition) (1)

if $max\{m, r\} = r$, then $min\{m, r\} = m$ (from (1)) (2)

if $max\{m, r\} = m$, then $min\{m, min\{m, r\}\} = min\{m, m\} = m$ (3)

Thus, $k = m\ \forall x$(from 1-3)

In particular, $A_{pt} \sqcap (A_{pt} \sqcup B_{pt}) = A_{pt}$ (by definition)

Proposition 3.8 (Idempotent laws of mpoint)

For any $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. Then

i. $A_{pt} \sqcup A_{pt} = A_{pt}$

ii. $A_{pt} \sqcap A_{pt} = A_{pt}$

Proof

i. Let $P_x^k \in A_{pt} \sqcup A_{pt}, P_x^l \in A_{pt}$ such that $k = max\{l, l\} = l$ (by definition)
Thus, $P_x^k = P_x^l$ (since $k = l\ \forall x$)
Hence, $A_{pt} \sqcup A_{pt} = A_{pt}$

ii. Let $P_x^k \in A_{pt} \sqcap A_{pt}, P_x^l \in A_{pt}$ such that $k = min\{l, l\} = l$ (by definition)
Thus, $P_x^k = P_x^l$ (since $k = l\ \forall x$)
Hence, $A_{pt} \sqcap A_{pt} = A_{pt}$

Proposition 3.9

For any $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. Then

i. $(A_{pt} \sqcap B_{pt})^* = A_{pt}^* \cap B_{pt}^*$

ii. $(A_{pt} \sqcup B_{pt})^* = A_{pt}^* \cup B_{pt}^*$

iii. $(A_{pt} \sqcap B_{pt})^n = A_{pt}^n \sqcap B_{pt}^n$

iv. $(A_{pt} \sqcup B_{pt})^n = A_{pt}^n \sqcup B_{pt}^n$ where $n \geq 0$

Proof

i. Let $P_x^l \in (A_{pt} \sqcap B_{pt})^*$ and $Q_{pt} = A_{pt} \sqcap B_{pt}$
$Q = MS(Q_{pt}) = MS(A_{pt} \sqcap B_{pt}) = A \Cap B$ (Proposition 2,2,15)
Since $P_x^l \in (A_{pt} \sqcap B_{pt})^*$, then $x \in Q^*$(by definition)
But $Q^* = (A \Cap B)^\star = A^* \cap B^*$ (Yager (1986))
Thus $x \in A^* \cap B^*$. i.e $x \in A^* \land\ x \in B^*$
In particular, $P_x^l \in A_{pt}^*$ and $P_x^l \in B_{pt}^*$. i.e $P_x^l \in A_{pt}^* \cap B_{pt}^*$
Hence, $(A_{pt} \sqcap B_{pt})^* \subseteq A_{pt}^* \cap B_{pt}^*$ (1)
Now let $P_y^l \in A_{pt}^* \cap B_{pt}^*$. We have $P_y^l \in A_{pt}^*$ and $P_y^l \in B_{pt}^*$
But $P_y^l \in A_{pt}^* \to P_y^r \in A_{pt}$ and $P_y^l \in B_{pt}^* \to P_y^m \in B_{pt}$

DOI: 10.56892/bima.v8i3A.811

Let $k = min\{r, m\}$. Then $P_y^k \in A_{pt} \sqcap B_{pt}$.

In particular, $P_y^l \in (A_{pt} \sqcap B_{pt})^*$ and $A_{pt}^* \cap B_{pt}^* \subseteq (A_{pt} \sqcap B_{pt})^*$ (2)

Comparing (1) and (2) above, the result follows

ii. Let $P_x^l \in (A_{pt} \sqcup B_{pt})^*$(by definition)

Let $Q_{pt} = A_{pt} \sqcup B_{pt}$

We have $Q = MS(Q_{pt}) = MS(A_{pt} \sqcup B_{pt}) = A \uplus B$

Since $P_x^l \in (A_{pt} \sqcup B_{pt})^*$, then $x \in Q^*$(by definition)

Where $Q^* = (A \uplus B)^* = A^* \cup B^*$ (Yager (1986))

Thus, $x \in A^* \cup B^*$. i.e $x \in A^* \vee x \in B^*$

In particular, $P_x^l \in A_{pt}^*$ or $P_x^l \in B_{pt}^*$. i.e $P_x^l \in A_{pt}^* \cup B_{pt}^*$

Hence, $(A_{pt} \sqcup B_{pt})^* \subseteq A_{pt}^* \cup B_{pt}^*$ (3)

Now let $P_y^l \in A_{pt}^* \cup B_{pt}^*$. We have $P_y^l \in A_{pt}^*$ or $P_y^l \in B_{pt}^*$

But $P_y^l \in A_{pt}^* \to P_y^r \in A_{pt}$ and $P_y^l \in B_{pt}^* \to P_y^m \in B_{pt}$

Let $k = max\{r, m\}$. Then $P_y^k \in A_{pt} \sqcup B_{pt}$.

In particular, $P_y^l \in (A_{pt} \sqcup B_{pt})^*$ and $A_{pt}^* \cup B_{pt}^* \subseteq (A_{pt} \sqcup B_{pt})^*$ (4)

Comparing (3) and (4) above, the result follows

iii. Let $P_x^k \in (A_{pt} \sqcap B_{pt})^n, P_x^r \in A_{pt} \sqcap B_{pt}, P_x^l \in A_{pt}, P_x^m \in B_{pt}$

such that $k = r^n$ and $r = min\{l, m\}$ (by definition)

In particular, $k = (min\{l, m\})^n$

But $(min\{l, m\})^n = min\{l^n, m^n\}$

Thus, $k = (min\{l, m\})^n = min\{l^n, m^n\} \forall x$

In particular, $(A_{pt} \sqcap B_{pt})^n = A_{pt}^n \sqcap B_{pt}^n$

iv. Let $P_x^k \in (A_{pt} \sqcup B_{pt})^n, P_x^r \in A_{pt} \sqcup B_{pt}, P_x^l \in A_{pt}, P_x^m \in B_{pt}$

such that $k = r^n$ and $r = max\{l, m\}$ (by definition)

In particular, $k = (max\{l, m\})^n$

But $(max\{l, m\})^n = max\{l^n, m^n\}$

Thus, $k = (max\{l, m\})^n = max\{l^n, m^n\} \forall x$

In particular, $(A_{pt} \sqcup B_{pt})^n = A_{pt}^n \sqcup B_{pt}^n$

Proposition 3.10

For any $A_{pt}, B_{pt}, C_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. The operation $\odot$ is distributive over the operation $\oplus$ i.e
$A_{pt} \odot (B_{pt} \oplus C_{pt}) = (A_{pt} \odot B_{pt}) \oplus (A_{pt} \odot C_{pt})$

Proof

$P_x^k \in A_{pt} \odot (B_{pt} \oplus C_{pt}), P_x^l \in A_{pt}, P_x^m \in B_{pt}, P_x^r \in C_{pt}$, such that $k = l.(m + r)$ (by definition)

But $k = l.(m + r) = (l.m) + (l.r)$ (distributivity of multiplication over addition)

Hence, $A_{pt} \odot (B_{pt} \oplus C_{pt}) = (A_{pt} \odot B_{pt}) \oplus (A_{pt} \odot C_{pt})$

Proposition 3.11.

DOI: 10.56892/bima.v8i3A.811

Let $A_{pt}, B_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. Then $A_{pt}\overline{\Delta}B_{pt} = (A_{pt} \ominus B_{pt}) \sqcup (B_{pt} \ominus A_{pt})$

Proof

Let $P_x^k \in A_{pt}\overline{\Delta}B_{pt}, P_x^m \in A_{pt}, P_x^l \in B_{pt}$

Clearly, $k = |m - l| = \begin{cases} m - l \text{ if } m - l > 0 \\ l - m \text{ if } m - l < 0 \end{cases}$ (by definition) (1)

Thus, $k = |m - l| = max\{m - l, l - m\}$ (from (1)) (2)

In particular, $A_{pt}\overline{\Delta}B_{pt} = (A_{pt} \ominus B_{pt}) \sqcup (B_{pt} \ominus A_{pt})$(by definition and (2))

Proposition 3.12

Let $A_{pt} \in \mathfrak{M}(\mathcal{S}_{\boldsymbol{pt}})$. Then $A_{pt}^0 = A_{pt}^*$

Proof

Let $P_x^k \in A_{pt}^0$ and $P_x^r \in A_{pt}$

We have $k = r^0 = 1$(Definition 2.2.12)

in particular $A_{pt}^0 = A_{pt}^*$.

CONCLUSION

This paper explores the fundamental operations on collections of finite mpoints, as introduced by Das and Roy (2021). We investigate the basic algebraic properties of these operations, including the application of De Morgan's laws to the complementation of union and intersection of mpoint collections. Furthermore, we presents the root set of mpoint collections under union and intersection operations. Our study provides a comprehensive understanding of the algebraic behavior of mpoints, contributing to the development of this emerging field.

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