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Moufang Loop of Odd Order pq^4r^3 and Associator Subloop of Odd Order r^2

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ABSTRACT

Under the condition that $q \not\equiv 1(mod\, p)$, with $3 < p < q$, where p and q are distinct odd primes, Moufang loops of order p^4q^3 has been shown to be associative (Rajah & Ademola, 2017). In this paper, we searched through the unique characteristics of Moufang loop of odd order pq^4r^3 and it's associator subloop of odd order r^2 under the conditions that p, q and r be odd primes with $p < q < r, q \not\equiv 1\ (mod\, p), r \not\equiv 1\ (mod\, p)$ and $r \not\equiv 1\ (mod\, q)$. This paper has also shown that Moufang Loop of odd order pq^4r^3 cannot have an associator subloop of odd order r^2.

Keywords: Moufang loop, Maximal subloop, Associator Subloop, Order, Nonassociative

INTRODUCTION

A loop G is said to be a Moufang loop if for any a, b, c in G the identity *(a·b)·(c·a)=(a·(b·c)) ·a.* is satisfied. It has been shown that once the order of a Moufang loop is known, then the Moufang loop can be classified and resolved.

The classification of even-order Moufang Loop have been completed by Chein, Leong and Rajah. The order Moufang Loop are still being resolved. Moufang Loop. Moufang loops of order 3^4 (Bruck, 1971) and p^5 for any prime $p > 3$ (Wright, 1965) which are nonassociative are known to exist. Also (Rajah, 2001) constructed a class of nonassociative Moufang loops of odd order pq^3 with the necessary condition $q \equiv 1(mod\, p)$. (Kanevsky, 2023, pp. 5-6) also proves the nonassociative inequality $(Q_0 + Q_1) + Q_2 \neq Q_0 + (Q_1 + Q_2) mod p^3$ which is a nonassociative composition example $mod p^3$.

One of the most important theorems in the study of Moufang loops would be Moufang's theorem: If there exist three (fixed) elements a, b, c in a Moufang loop that associate in some order, then these elements generate a group. As a corollary, Moufang loops are diassociative, i.e. for any two (fixed) elements a and b in a Moufang loop, they generate a group. Moufang loops need not be associative since there exists a nonassociative Moufang loop of order 12 (Chein, 1974).

Moufang loops are a fundamental concept in abstract algebra, playing a crucial role in the study of nonassociative structures. The classification of Moufang loops of small orders is an active area of research, with significant implications for our understanding of algebraic symmetries. The specific case of Moufang loops of order p^4q^3, where p and q are odd primes, presents an intriguing problem due to the complexity of their associator subloops.

This study aims to contribute to the ongoing efforts to classify Moufang loops of small orders by investigating the unique characteristics of Moufang loops of order pq^4r^3 and their associator subloops. By exploring the conditions under which these loops are associative, we seek to deepen our understanding of the algebraic structures that govern their behavior. Furthermore, the discovery that Moufang loops of odd order cannot have an associator subloop of odd order

r^2 has significant implications for the broader study of non-associative algebraic structures. This research has the potential to shed new light on the fundamental principles governing these structures and to inform further research in this area.

Definitions and Notations

1) A loop $\langle G,\bullet\rangle$ is a binary system that satisfies the following two conditions:

(i) Specification of any two of the elements a, b, c in the equation $a.b = c$ uniquely determines the third element, and

(ii) The binary system contains an identity element e such that $ea = ae = a$ for all a in G.

2) For any $a, b, c \in G$, a moufang loop is a loop $\langle G,\bullet\rangle$ such that $(a.(b.c)).a = (a.b).(c.a)$.

(We shall from now on, for sake of briefness simply refer to the loop $\langle G,\bullet\rangle$ as the loop G. also, we shall write $(a.b).c$ simply as $ab.c$, $(a.(b.c)).a$ as $(a.bc)a$, etc).

3) In Moufang loop, the associator sub-loop $Ga = (G, G, G) = \langle(g_1, g_2, g_3)|g_i \in G\rangle$. In a Moufang Loop, G_a is the sub-loop generatd by all the associators (a, b, c) such that $(a,b,c) = (a.bc)^{-1}(ab.c)$. Clearly G is associative if and only if G_a=1

4) An elementary abelian group (or elementary abelian p-group) is an abelian group in which every non-trivial element has order p. The number p must be prime and the elementary abelian groups are a particular kind of $p-$group. The case where $p = 2$, i.e. an elementary abelian 2-group, is sometimes called a Boolean group.

5) Let D be a group and let F be a normal subgroup of D. then $D/F = \{dF: dF\}$ is the set of all cosets of F in D and is called the quotient group of F in D.

Intuitively we say $D/F = all\ elements\ in\ D\ that\ are\ not\ in\ F.$

6) The order of a group (D) is the number of elements present in that group, i.e. its cardinality. It is denoted by $|D|\ or\ o(D)$.

7) $I(G) = (R(a,b), G(a,b), T(a)|a,b \in L)$ is called the inner mapping group of D, where

$wR(a,b) = (ca.b)(ab)^{-1}$

$wG(a,b) = (ba)^{-1} (b.ac)$

$wT(a) = a^{-1}.ca.$

8) The subloop generated by all $n \in G$ such that $(n, a, b) = (a, n, b) = (a, b, n) = 1$ for any $a, b \in G$ is said to be the nucleus of G. This is denoted as $N(G)$ or just as N.

9) Assume U is a subloop of G. Then

$CL\ (U) = (\ ul = l\ G|lu)$ for all $u \in U$.

10) Let π a set of primes and V be a subloop of G. Then:

(a) G has a normal subloop V, denoted as $V \lhd G$, if $V\theta = V$ for all $\theta \in I\ (G)$.

(b) A nonnegative integer n is a π-number if every prime divisor of n lies in π.

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(c) We let n_π be the largest π-number that divides n for each nonnegative integer n.

(d) V is a π-loop if the order of every member of V is a π-number.

(e) If $|V| = |G|$. Then V is a Hall -subloop of G.

(f) V is a Sylow p-subloop of G if V is a Hall -subloop of G and contains only a single prime p.

9. Assume R is a normal subloop of G.Then:

(a) R is a proper normal subloop of if G if $R \neq G$.

(b) G/R is a proper quotient loop of G if $R \neq \{1\}$.

10. Suppose R is a normal subloop of G. Then:

(a) If V is non-trivial and contains no proper nontrivial subloop which is normal in G then R is a minimal normal subloop of G. In other words, if there exists $S \lhd L$ with $\{1\} < R < V$, then $R = \{1\}$ or V.

(b) If R is not a proper subloop of every other proper normal subloop of G, then R is a maximal normal subloop of G. In other words, there exists $R \lhd G$ such that $V < R$, then $V = R$ or G.

11. For any distinct integers r and s, (r, s) is known as the greatest common divisor of r and s.

12. Quasi-group: A set with a binary operation (usually called multiplication) in which each of the equations $xa = y$ and $bx = y$ has a unique solution, for any element $x.y$ of the set. A quasi-group with a unit is called a loop.

13. Sub quasi-group: A subset S of a quasi-group Q is a sub-quasi-group of Q if, when equipped with the same binary operation $*$ in Q for any elements x, y in S, there exists unique elements a, b in S such that $x * a = y$ and $b * x = y$.

Basic Properties and Known Results

Let G be a Moufang loop.

Lemma 2.1. G is dissociative, that is, $\langle a, b\rangle$ is a group for any $a, b\ in\ L$. Moreover, if $(a, b, c) = 1$ for some $a, b, c, in\ L$, then $\langle a, b, c\rangle$ is a group. (Bruch, 1958, p.117)

Lemma 2.2. $N = N(L)$ is a normal subloop of L. Clearly N is a group by its definition. (Bruch, 1958, p.114)

Lemma 2.3. Assuming $V \lhd G$, then;

(a) $G_a \subset V$ implies G/V is a group.

(b) $G_c \subset V$. implies G/V is commutative. (Leong & Rajah, 1996, p.563).

Note that the properties above hold for all Moufang loops in general. However, the following properties hold on for finite Moufang loops G.

Lemma 2.4. Suppose V is a subloop of G. Then $|V|$ divides $|G|$. (Grishkov & Zavarnitsine, 2005, p.42).

Lemma 2.5. Let G be a loop, V a subloop of G and π is a set of primes. Then

a) G is solvable. (Glauberman, 1968, p.413)

b) If V is a minimal normal subloop of G, then V is an elementary abelian group and $(V, V, G) = \langle v_1, v_2, g) | v_i \in v, g \in G\rangle = \{1\}$. (Glauberman, 1968, p.402)

c) V is a normal subloop of G, $(|G/V|, |V|) = 1$ and $\{1\} = (V, V, G)$ implies $V \subset N$. (Glauberman, 1968, p.405)

d) G contains a (Hall π-)subloop. (Glauberman, 1968, p.409).

Lemma 2.6. if G has any of the following orders, then G is a group:

a. For distinct primes p and q, $|G| = p, p^2, p^3$ *or* pq is a group. (Chein, 1974, p.35).

b. pqr or p^2q where p, q and r are distinct odd primes. (Leong & Rajah, 1995, p.269).

c. p^4 where p is a prime and $p > 3$. (Purtill, 1988, p.33)

d. pq^2 where p and q are distinct odd primes. (Leong, 1974, p.124 and p. 6).

e. $p_1p_2...p_mq^3r_1r_2...r_n$ with $p_1 < p_2 < ... < p_m < q < r_1 < r_2 ... < r_n$ *and* $q \not\equiv 1 \pmod{p_i}$ for all $I \in \{1, 2, ... m\}$, where $p_1, p_2..., p_m, q, r_1, r_2, ... ,r_n$ are odd primes . (Rajah & Chee, 2011, p.374)

f. $p^\alpha q_1 \dots q_n$, where $\alpha \leq 3$ *and* $p, q_1, ..., q_n$, are distinct odd primes with $p < q_i$. (Leong & Lim, 1994, p.350).

g. $p^\alpha q_1 \dots q_n$, where $\alpha \leq 4$ *and* $p, q_1, ..., q_n$, are distinct odd primes with $3 < p < q_i$. (Leong & Rajah, 1996, p.567).

h. $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$, where $1 \leq \alpha_i \leq 2$ *and* $p_1, p_2... ,p_n$ are distinct odd primes. (Leong & Rajah, 1996, p.882).

i. $p^\alpha q_1^{\beta_1} q_2^{\beta_2} \dots q_n^{\beta_n}$, with $\alpha \leq 3$ when $p > 2$, or $\alpha \leq 4$ when $p > 3$, where p and q_i are primes with $p < q_1 < ... < q_n$ and $\beta_i \leq 2$. (Leong & Rajah, 1997, p.483).

j. $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n} q^3$, with $q \not\equiv 1 \pmod{p_i}$ and $1 \leq \alpha_i \leq 2$, where $p_1, p_2, ... ,p_n$ and q are distinct odd primes. (Rajah & Chee, 2011, p.970).

k. p^3q^3, where p and q are odd primes with $p < q$, and $q \not\equiv 1 \pmod{p}$. (Rajah & Chee, 2011, p.1364).

l. pq^4, where p and q are odd primes with $p < q$, and $q \not\equiv 1 \pmod{p}$. (Chee & Rajah, 2014, p.434)

m. p^4q^3 where p and q are distinct odd primes. (Rajah & Ademola, 2016)

n. $p_1^3 \dots p_n^3$ where p and q are distinct odd primes. (Ademola & Rajah, 2016, pp.1400-1404)

o. $p^4 q_1^3 q_2^3 \dots q_n^3$, with $3 < p < q_1 < q_2 ... < q_n$, $q_i \not\equiv 1 \pmod{p}$ and $q_i \not\equiv 1 \pmod{q_j}$, where p and each q_i are primes. (Rajah & Ademola, 2017)

Lemma 2.7. Suppose the order of G is odd and every proper subloop of G is a group. If there exists a minimal normal Sylow subloop in G, then G is a group. (Leong & Rajah, 1995, p.268).

Lemma 2.8. If there exist U, V in G such that $(|V/U|, |U|) = 1$ *and* $U \lhd V \lhd G$. Then $U \lhd V$. (Leong & Rajah, 1995, p.879)

Lemma 2.9. Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n} q^{\beta}$, where $1 \leq \beta \leq 2$ and each p_i is a distinct odd prime such that $p_i < q$. Suppose:

i. Every proper subloop of G is a group, and

ii. There exists a Sylow q-subloop normal in G.

The L is a group. (Leong & Rajah, 1995, p.879)

Lemma 2.10. Let G be a minimal normal subloop of G, where G is of odd order such that $U \subset G$, and Q, a hall subloop of G. Assume that all proper quotient loops and all proper subloop of G are groups. $(|U|, |Q|) = 1$ and $Q \lhd UQ$. Then G is a group. (Leong & Rajah, 1995, p.564)

Lemma 2.11. Let the order of G be odd such that every proper subloop and quotient loop of G is a group. Assume Q is a Hall subloop of G such that $(|G_a|, |Q|) = 1$, and $Q \lhd G_a Q$. Then G is a group. (Leong & Rajah, 1995, p.564).

Lemma 2.12. Let G be nonassociative and of odd order such that all proper quotient loops of

G are groups. Then:

a) G_a is a minimal normal subloop of G; and is an elementary abelian group. (Rajah & Chee, 2011, p.478) and (Glauberman, 1968, p.402).

b) If U is a maximal normal subloop of G, then G_a and G_c lie in U. Furthermore, $G = U\langle a \rangle$ for any $a \in G \setminus U$. (Rajah & Chee, 2011, p.478).

Lemma 2.13. Suppose

a. $|G| = p^{\alpha} m$ where p is a prime, $(p, m) = (p - 1, p^{\alpha} m) = 1$ and G has an element of order p^{α}. Then there exists a (Sylow p-)subloop p of order and a normal subloop U of order m in G such that $G = PU$. (Leong & Rajah, 1998, p.39).

b. $|G| = p^2 m$ where p is the smallest prime dividing $|G|$ and $(p, m) = 1$. Then there exists a subloop P of order p^2 and a normal subloop U of order m in G such that $G = PU$. (Leong, 1976, p.33).

Lemma 2.14. Let V be a normal Hall subloop of G, where G is of odd order. Assume $V = < a > G_a$ for some $a \in V \setminus G_a$ and $G_a \subset N$. then $V \subset N$. (Rajah & Chee, 2011, p.17).

Lemma 2.15. Let G be of odd order and nonassociative, and let U be a maximal normal subloop of G. Suppose all quotient loops and proper subloops of G are groups. Then;

a. G_a is a Sylowsubloop of N, then $G_a = N$. (Rajah & Chee, 2011, p.480).

b. G_a is cyclic, then $G_a \subset N$. (Leong, 1976, p.480).

c. $(v, t, g) = 1$ for all $v \in G_a, t \in U, g \in G$, then $G_a \subset N$. (Leong, 1976, p.479)

d. $(v, t, g) \neq 1$ for some $v \in G_a, t \in U, g \in G$, then G_a contains a proper nontrivial subloop which is normal in U. (Leong & Rajah, 1995, p.19).

Lemma 2.16. Suppose every proper subloop of G is a group and $|G|$ is odd. If N contains a Hall subloop of G, then G is a group. (Leong & Rajah, 1995, p.564)

Lemma 2.17. Let G be of order $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n}q$, with $p_1 < p_2 < \dots < p_n < q$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbf{Z}^+$, $q \not\equiv 1 \pmod{p_i}$ for all i, where $p_1, p_2 \dots, p_n$ and q are odd primes. Then there exists a normal subloop of order $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n}$ in G. (Rajah & Chee, 2011, p.1362).

Lemma 2.18. Let G be of odd order and nonassociative, and let U be an associative maximal normal subloop of G and Q be a subloop of G. Assume $G_a \subset N$ and $U = G_aQ$. Then for any $a \in G \setminus U$, there exist some $u, v \in Q$ such that $(a, u, v) \neq 1$. (Rajah & Chee, 2011, p.1362).

Lemma 2.19. Assume p and q are distinct odd primes. If and only if $q \equiv 1 (mod\ p)$, then there exists a nonassociative Moufang loop of order pq^3. (Rajah, 2001, p.78 and p. 86).

Lemma 2.20. Let G be a Moufang loop of odd order p^3m with $(p, m) = 1$ and p is the smallest prime divisor of G. Assume there exists T a normal subloop of order p or p^2 in G. then there exists a subloop of order m normal in G. (Ademola & Rajah, 2016).

Lemma 2.21. Let L be a nonassociative Moufang loop of order $p^\alpha m$ where p is prime and $(p, m) = 1$, such that all proper quotient loops and proper subloops of G are groups. Assume $|G_a| = p^{\alpha-1}$ and $G_a \subset N$. Then $G_a = N = C_G(N)$. (Ademola & Rajah, 2016, p.1400)

Lemma 2.22. Let G be a nonassociative Moufang loop of order $p_1p_2^2q^4$, where p_1, p_2, q are distinct odd primes with $p_i < q$, $q \not\equiv 1 (mod\ p_i)$ and $p_2 \not\equiv 1 (mod\ p_1)$. Assume all proper subloops and proper quotient loops of G are groups and $(|G|, 6) = 1$, *then* $|G_a| = q^2$. (Yunus, 2023, p.18).

MAIN RESULTS

Theorem 4.1

Let G be a nonassociative moufang loop of odd order pq^4r^3 where p, q and r are odd primes with $3 < p < q < r, q \not\equiv 1 \ (mod\ p), r \not\equiv \ (mod\ p)$ and $r \not\equiv 1 \ (mod\ q)$. Then the following hold:

(a) G has finite possible divisor

(b) The subloops and quotient loops of (a) above are associative for each order

(c) G is solvable

Proof:

(a) The possible divisors of pq^4r *are: p, q, r,* q^2, q^3, q^4, pq, pq^2, pq^3, pq^4, rq, rq^2, rq^3, rq^4, pqr, pq^2r, pq^3r, pq^4r, *1* and pr.

(b) (i) If the subloopor quotient is of order p, q, r, q^2, q^3 then by Lemma 2.6(a), the subloop or quotient loop is associative.

(ii) If the subloop or quotient is of order pq^2 then by Lemma 2.6(d), the subloop or quotient loop is associative.

(iii) By Lemma (2.6)(b), if the subloop or quotient loop is of order pqr, then the subloop or quotient loop is associative.

(iv) If the subloop or quotient loop is of order pq^4, then by lemma (2.6)(l) the subloop or quotient loop is associative.

(c) If the order of a Moufang loop is finite, we see by lemma 2.5(a) it is solvable. Therefore, we conclude that only finite Moufang loops are solvable. Yet the Moufang loop of order pq^4r is solvable because the order is finite.

Theorem 4.2.

Let G be a nonassociative moufang loop of odd order pq^4r^3 where p, q and r are odd primes with $3 < p < q < r, q \not\equiv 1\,(mod\,p), r \not\equiv \ (mod\,p)$ and $r \not\equiv 1\,(mod\,q)$, such that all proper subloops and proper quotients loops of G are associative. Then G has a finite possible order of the associator loop G_a.

Proof:

Given that the Moufang loop is of order $pq^4r^3, p < q < r, p \not\equiv 1\,(mod\,q), p \not\equiv 1\,(mod\,r)$ and $r \not\equiv 1\,(mod\,q)$, by lemma (2.5)(b), G_a is an elementary abelian group such that all proper subloops and quotient loops of G are associative. Therefore, the possible order of the associator subloop G_a are: $q^4, q^3, q^2, q, p, r, r^2, r^3$.

Theorem4.3.

Given a non associative Moufang loop G of odd order pq^4r, with $p < q < r$ and $q \not\equiv 1(mod\,p), r \not\equiv 1(mod\,p)$ and $r \not\equiv 1\,(mod\,q)$. Then the order G_a cannot be p, q or q^4.

Proof

Given G a non associative Moufang loop, we see by Lemma (2.5)(a) that G is solvable. Also by Lemma (2.12), G_a is minimal normal subloop of G. then by Lemma (2.5)(b), G_a is an elementary abelian group since all proper subloops and quotient loops of G are associative.

So, $|G_a| = p, q, q^2, q^3, q^4$, or r.

If $|G_a| = p, q^4$ or r, then G_a would be a minimal normal Sylow subloop of G and so by Lemma (2.7), G would be a group which is a contradiction to the fact that G is a nonassociative Moufang Loop.

Case 1: $|G_a| = q$

By Lemma (2.5)(d), there exists Q a Hall subloop of order p^4 in G. Since $G_a \lhd G$, $G_a Q < G$. So, $|G_aQ| = \frac{|G_a||Q|}{|G_a \cap Q|} = pq$ or $|G_aT| = \frac{|G_a||T|}{|G_a \cap T|} = tq$. Since Q is a Sylow p-subloop of $G_a Q$, by Lemma (2.19), $Q \lhd G_a Q$. Also, since $(|Q|, |G_a|) = (p^4, q) = 1$, by Lemma (2.10), G is a group.

Therefore, when $|G_a| = q, q^4$ or r, it contradict the Moufang loop G being non associative.

Theorem4.4

Let G be a nonassociativemoufang loop of odd order pq^4r^3, where p, q and r are odd primes with $3 < p < q < r, q \not\equiv 1\,(mod\,p), r \not\equiv \ (mod\,p)$ and $r \not\equiv 1\,(mod\,q)$, such that all proper subloops and proper quotients loops of G are associative. Then G_a cannot have an odd order r^2.

Proof:

Suppose G is nonassociative. By Lemma (2.5)(a) G is solvable.

By Lemma (2.12)(a) G_a is normal in G. We know by Lemma (2.19) that there exists a maximal normal subloop of order pq^4r^2 in G and by Lemma (2.8) there exist K a Sylow subloop of order pq^4 in G. Where K is normal in G.

Since K is normal hall in G. It follows that G/K is a group. Hence by Lemma (2.2), we get that G_a is contained in K.

Now by Lemma (2.4) and Langrange's theorem, since G_a is contained in K, it follows that the order of G_a must divide the order of K.

So $|G_a| = r^2$ divides $|K| = pq^4$. Clearly, this is impossible as p, q and r are primes and

$(|G_a|, |K|) = (r^2, pq^4) = 1$.

So if G satisfies the condition given, it follows that G_a cannot have an odd order r^2.

This concludes the proof.

Theorem 4.5

Let G be a nonassociative moufang loop of odd order pq^4r^3 where p, q and r are odd primes with $3 < p < q < r, q \not\equiv 1\ (mod\ p), r \not\equiv\ (mod\ p)$ and $r \not\equiv 1\ (mod\ q)$, such that all proper subloops and proper quotients loops of G are associative and $|G_a| = r^2$, then there exists a maximal normal subloop T of order pq^4r^2 in G.

Proof:

Assuming that G is nonassociative. By Lemma (2.5)(a), G is solvable. By Lemma (2.12), $G_a \lhd G$. Therefore, $|G/G_a| = pq^4$r. By Lemma (2.19), there exists a subloop $T/G_a \lhd G/G_a$ such that $|T/G_a| = pq^4$. Therefore, $T \lhd G$. This implies that $|T| = pq^4r^2$. Now $|G|/|T| = r$. This shows that T is the maximal normal subloop of G.

Theorem4.6

Let G be a nonassociative Moufang loop of odd order pq^4r^3 where p, q and r are odd primes with $3 < p < q < r, q \not\equiv 1\ (mod\ p), r \not\equiv\ (mod\ p)$ and $r \not\equiv 1\ (mod\ q)$, such that all proper subloops and proper quotients loops of L are associative. Then

(i) There exist $U \lhd V \lhd T$,

(ii) $U \lhd G$ whenever $U \lhd V \lhd T \lhd G$ and $|U| = pq^4$

Proof:

By Theorem 4.5 $|T| = pq^4r^2$,

(i) From Lemma (2.17)(d), if $(v, t, g) \neq 1$ for some fixed $v, t, g \in G$ with $v \in G_a$ and $t \in H$.

Then G_a contains Q a proper nontrivial subloop normal in T. Thus $|Q| = r$, so $|T/Q| = pq^4r$. Then by lemma (2.19), there exists $V/Q \lhd T/Q$ such that $|V/Q| = pq^4$, so $V \lhd T\ and\ |V| = pq^4r$. Again, there exists $U \lhd V$ such that $|U| = pq^4$. Since U is normal Hall subloop in V by

Lemma (2.8) $U \lhd T$. By the same Lemma (2.8) we see that $U \lhd V \lhd T$ and $(|U|, |V/U|) = (pq^4, r) = 1$.

(ii) We know $|T| = pq^4r^2$ and $|U| = pq^4$, thus U is again a sylow subloop of T. By lemma (2.8) $U \lhd T \lhd G$ and $(|U|, |T/U|) = (pq^4, r^2) = 1$. This implies that $U \lhd G$.

CONCLUSION

This paper has made a further investigation on the order of associator subloops of the Moufang loop of odd order pq^4r^3 with the conditions that p, q and r be primes with $p < q < r$, $q \not\equiv 1\ (mod\ p)$, $r \not\equiv 1\ (mod\ p)$ and $r \not\equiv 1\ (mod\ q)$, obtaining all possible orders of the asosciator subloop and focus on $|G_a| = r^2$.

The paper also marks a significant advance in unraveling the mysteries of Moufang loops of odd order. As we conclude this study, we acknowledge the complexity and richness of these mathematical structures, and we hope that our findings will inspire and guide future researchers in exploring this fascinating field.

7 Recommendation

The paper is still open for further investigations as future research could explore the properties and structures of the associator suloops of the Moufang loop of odd order pq^4r^3.

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