Bima Journal of Science and Technology, Vol. 8(3) Sept, 2024 ISSN: 2536-6041

DOI: 10.56892/bima.v8i3.756

Characterization of Subnormal Operators in C*-algebra

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ABSTRACT

The study investigates subnormal operators in C*-algebras, addressing unresolved challenges in their characterization. This research aims to provide a clear algebraic framework for subnormal operators, utilizing matrix representations to explain their properties of normality, subnormality, self-adjointness, and isometry. The specific objectives include stating the algebraic characterization of subnormal operators, offering concise characterizations of C*-algebra elements, and establishing results from operator theory that highlight the invertibility and other critical properties of these elements. Grounded on Muhly and Solel, Rørdam and Williams theorem, the methodology involves a theoretical analysis of subnormal operators in Hilbert spaces and their relationships with C*-algebras. The research findings successfully achieve the set objectives, presenting a detailed algebraic characterization of subnormal operators and elucidating their role within C*-algebras. The conditions for subnormality were effectively applied, improving the understanding of the structure and dynamics of these algebras.

Keywords: *C*-algebra, normal operators, subnormal operators, Hilbert space*

INTRODUCTION

A C*-algebra is a mathematical structure that combines algebraic operations and a concept of “closeness” in a way that preserves essential properties of complex numbers. It is a type of algebra where element can be added, multiplied and have a notion of adjoint (like complex conjugation) and the algebra also has a norm that behaves well with these operations. Karthiks. L. (2022). The characterization of subnormal operators in the context of C*-algebras has been an active area of research in operator theory, with significant contributions from various authors in recent years. Subnormal operators are a special class of operators that generalize the concept of normal operators and have been studied extensively for their rich mathematical structure and applications.

Ando and Watatani, (2018) investigated the connection between subnormal operators and the theory of C*-algebras. They provided a comprehensive characterization of subnormal operators in terms of the existence of a normal extension within a larger C*-algebra. Specifically, they showed that an operator T in a C*-algebra A is subnormal if and only if there exists a Hilbert space H and a normal operator N $\in$ B(H) such that A can be isometrically embedded as a C*-subalgebra of B(H), and T is the restriction of N to the subspace A.

Rørdam, M. (2020), studied the spectral properties of subnormal operators in C*-algebras. He established a functional calculus for subnormal operators, which allowed for the analysis of their spectral behavior within the C*-algebraic framework. Katsoulis (2014) provided a deeper understanding of the interplay between the operator-theoretic and C*-algebraic aspects of subnormal operators.

More recently, Muhly and Solel (2021) explored the connections between subnormal operators and the theory of operator algebras. They investigated the relationship between subnormal operators and the notion of hereditary C*-subalgebras, providing new insights into the structure of subnormal operators in the C*-algebraic setting.

PRELIMINARIES AND NOTATION

Definition: Katsoulis, E. (2014)**:** An algebra A is called a $*$ - algebra if it is complex algebra with conjugate linear involution $*$ which is anti-isomorphism, i.e for any $a, b \in A$ and $\alpha \in C$,

$$(a+b)^* = a^* + b^*, \ (aa)^* = \overline{a}a^*, \ a^{**} = a \ \ and \ \ (ab)^* = b^* a^*.$$

If $a \in A$ is called the adjoint of a. Let A be a $*$-algebra which is also normed algebra. A norm on A that satisfies

$$\left\|a^* a\right\| = \|a\|^2$$

For all $a \in A$ is called a $C^* - norm$. If, with this, A is complete, then A is called a $C^* -$algebra.

Definition: Murphy, G. (2021): A (bounded linear) operator T on a (complex) Hilbert space $\mathfrak{H}$ is called normal if T commutes with its adjoint, T*.

Definition: Ando & Watatani (2018): Let $\mathfrak{H}$ be a separable Hilbert space over the complex field $\mathbb{C}$ and let $L(\mathfrak{H})$ be the collection of all linear bounded operators on $\mathfrak{H}$. An operator S in $L(\mathfrak{H})$ is called a subnormal operator if there is a Hilbert space K containing $\mathfrak{H}$ and there is a normal operator N on K which leaves $\mathfrak{H}$ invariant so that N restricted to $\mathfrak{H}$ is S.

METHOD

The following theorems provide detailed support for the methodology, ensuring an effective and comprehensive foundation for its application and analysis.

Theorem 1 (Muhly and Solel, 2021): Suppose that S is a subnormal operator in $L(H)$. Then there exist a normal operator N and an isometry W in $L(H)$ such that $S = W^* NW$ and $NP = NP$, where $P = WW^*$.

Theorem 2 (**Rørdam, 2020**): For every C*-algebra A, there exist a Hilbert space H and a *-isomorphism $\pi = A \to B(H)$ where $B(H)$ denote the algebra of all bounded operators on H

Theorem 3 (**L. R Williams 1988**): Suppose that A is a C*-algebra that has a complementary pair of isometries and is dual closed, and suppose that S belongs to A. Then S is subnormal if and only if there exist a normal N and an isometry W in A such that $S = W^* NW$ and $NP = PNP$, where $P = WW^*$. Furthermore, if W is any isometry in A that has a complement in A and if S is a subnormal element of A, then there exists a normal element N of A such that $S = W^* NW$ and $NP = PNP$, where $P = WW^*$.

RESULTS

1. Let A be a C*-algebra and $H \in A$. Assume the existence of a normal element N and an isometry W in A such that $S = W^*NW$ and $NP = PNP$, where $P = WW^*$. Under these conditions, it follows that S is a subnormal element of A.

Proof:

In other to prove this, we first need to show that it satisfies the properties of C*-algebra, namely closure under addition, scalar multiplication, and the adjoint operation, well as being closed under the norm defined by the inner product, as stated in 2 below.

Using the method of Murphy, G. (2021).

2.

- A is a C*-algebra.
- N is a normal element in A.
- W is an isometry in A.
- is a projection in A. $P = WW^*$

We define $S = W^*NW$.

(a) Addition and Scalar Multiplication: Since A is a C*-algebra, it is closed under addition and scalar multiplication. S is expressed as a product of elements in A, namely $W^*, N, and\ W$, all of which are assumed to belong to A. Hence, S is in A by closure under multiplication and addition.

(b) Adjoint Operation: Let's compute the adjoint of S, denoted S^*. Using the properties of the adjoint operation and the properties of $W\ and\ N$ we have:

$$S^* = (W^*NW)^* = W^*N^*(W^*)^* = W^*N^*W = S$$

Thus, S is self-adjoint.

(c) Closed under Norm: To show that S is closed under the norm, we need to verify that $\|S\| = \|W^*NW\|$ is finite. Since W is an isometry, it preserves the norm, i.e.,. Thus, $\|S\| = \|W^*NW\| = \|N\|$ which is finite because N is assumed to be bounded.

Therefore, S satisfies the properties of a C*-algebra element, and hence, it belongs to the C*-algebra A, now we proceed with the prove of the problem.

From Theorem (1),

1. **Proving** S is normal:
   Since N is normal, we have $NN^* \; N^*N$. Therefore, we have

$$
\begin{aligned}
SS^* &= (W^*NW)(W^*NW)^* \\
&= W^*NN^*W \\
&= W^*N^*NW \\
&= (W^*N^*W)(W^*NW) \\
&= S^*S
\end{aligned}
$$

   Thus, S is normal.
2. Proving $NP = PNP$:
   Given that $P = WW^*, P$ is a projection operator. We can observe that P is self-adjoint (i.e ., $P^* = P$) and $P^2 = P$.
   Now, let's compute $NP \; and \; PNP$

$$
\begin{aligned}
NP &= NW^*W^* \\
PNP &= WW^*NW^*W^* \\
&= W(W^*NW^*W^*)W^* \\
&= W\,N^*W^* \\
&= NW^*W^* \\
&= NP
\end{aligned}
$$

   Hence, $NP = PNP$.

   Since S is normal and $NP = PNP$, *by* definition, S is subnormal.

   Therefore, if A is a C*-algebra and $S \in A$ is given as $S = W^*NW$, where N is normal and W is an isometry in A, and if $NP = PNP$ holds for some projection $P = WW^{**}$, then S is a subnormal element of A.

2. Let A be a C*-algebra and let $T \in A$. The following properties characterize T.

i. **T is self-adjoint if and only if $T = T^*$**
ii. **T is normal if and only if $TT^* = T^*T$**
iii. **T is an isometry if and only if $T^*T = I$**

Proof:

Using theorem (1) we can show:

i. To prove that T is self-adjoint if and only if $T = T^*$, we need to show both directions of the equivalence.

DOI: 10.56892/bima.v8i3.756

First, recall from result 1 under (b). Assume that T is self-adjoint. This means that $T = T*$, which implies that the adjoint of T is equal to itself. Therefore, $T = T*$.

Conversely, suppose $T = T*$. We want to show that T is self-adjoint. Muhly and Solel (2021) define the adjoint of an operator as the unique operator satisfying $\langle Tx, y\rangle = \langle x, T*y\rangle$ for all x, y in the underlying inner product space. Now, let's consider the inner product $\langle Tx, y\rangle$ for any x, y in the inner product space. Using the assumption $T = T*$, we have $\langle Tx, y\rangle = \langle x, Ty\rangle$. Since $T = T*$, we can rewrite this as $\langle x, Ty\rangle = \langle x, Ty\rangle$ which holds true for all x, y in the inner product space. Therefore, T is self-adjoint.

Hence, we have shown that T is self-adjoint if and only if $T = T*$.

ii. From Result 1 using the method of [1] T is subnormal if and only if there exist a normal matrix N such that T is the restriction of N to an invariant subspace.

If T is subnormal: By definition, there exists a larger Hilbert space and a normal operator N (in matrix form) such that T is the restriction of N to an invariant subspace.

Conversely, if there exist a normal matrix N such that T is the restriction of N to an invariant subspace: By definition, T is subnormal.

iii. T is an isometry if and only if $T*T = I$

To prove that T is an isometry if and only if $T*T = I$,Using the method in Theorem [3] we once again need to establish both directions of the equivalence.

Assume that T is an isometry. This means that $T*T = I$, which implies that the product of the adjoint of T with T is equal to the identity operator.

Conversely, suppose $T*T = I$. We want to show that T is an isometry. Recall that an operator is an isometry if and only if its adjoint multiplied by itself yields the identity operator. Now, let's consider the product $T*T$.

Using the assumption

$$T*T = I ,$$

we have

$$T*T - I = 0$$

Rearranging, we get ,

$$T*T - I = 0$$

which can be further simplified as

$$(T*T - I)* = 0.$$

Expanding the adjoint, we have

$$T*T - I* = 0$$

T is an isometry if and only if $T*T = I$

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3. Let A be a C*-algebra and $\pi : A \to B(H)$ be a *-isomorphism. If $\pi(T)$ is invertible in $B(H)$, then T is invertible in A.

Proof:

Using the method in theorem [2]

Assume $\pi(T) \in B(H)$ is invertible. This means there exist an operator $\pi(S) \in B(H)$ such that $\pi(T)\pi(S) = \pi(S)\pi(T) = I_H$, where I_H is the identity operator on H.

Since π is a *-isomorphism, it is bijective. Thus, there exist a unique operator $S \in A$ such that $\pi(S)$ is the inverse of $\pi(T)$. This means

$$\pi(T)\pi(S) = \pi(S)\pi(T) = I_H.$$

Because π is a *-isomorphism, it preserves the multiplicative structure. Therefore, the fact that

$$\pi(T)\pi(S) = I_H \text{ and } \pi(S)\pi(T) = I_H$$

implies that $TS = ST = I_A$, where I_A is the identity element in A.

Therefore, We have shown that

$$TS = ST = I_A,$$

which means S is the inverse of T in A. Therefore, T is the inverse of A.

CONCLUSION

This paper illustrates the relationship between the algebraic structure of C*-algebras and the spectral theory of subnormal operators. It shows how the well-behaved nature of C*-algebras and the spectral properties of their elements can be leveraged to understand and extend the behavior of subnormal operators beyond their initial Hilbert space settings.

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