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Representation and Stability of Solutions for Systems of Delay Differential Equations With Multiple Constant Delays

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ABSTRACT

Delay differential equation is type functional differential equations that arise in numerous areas of applied sciences. Various forms of these equations play a vital role in mathematical modelling of real-life phenomena.Different techniques have been use to obtain solutions of various types of delay differential equations (DDEs). However, many situations arising in the theory of DDEs in which the solution of certain types of nonlinear DDEs cannot be explicitly obtained. In such a case, having a suitable representation for the solutions, some important properties such as oscillation and stability of these equations can be obtained. Therefore, in this work, a Natural transform and convolution theorem are used to derive a closed form-formula for the representation of solution for nonlinear systems of DDEs. The derived result is also used to study the exponential stability for the solution of nonlinear systems of DDEs. Hence, the approach can also be applied to study the stability analysis of many types of nonlinear problems.

Keywords: Natural transform, Convolution Theorem, Nonlinear System of DDEs.

INTRODUCTION

In a qualitative study of DDEs, many situations arising in which certain types of such equations cannot be explicitly solved. However, some features (oscillation, stability, etc) for the solution of those equations can be determined from it representation. Therefore, numerous researchers used different techniques to obtain a good representation of solutions to different types of DDEs, more especially the systems of such equations. Khusainove and Shuklin (2003) applied the method of step to construct the so-called delayed matrix exponential.

$$e_{\tau}^{Bt} = \begin{cases} \varnothing, & t < -\tau \\ I, & -\tau < t < 0 \\ I + Bt + \frac{B^2(t-\tau)}{2} + \dots & (k-1)\tau \le t < k\tau, k \in N \\ + \frac{B^k\left(t-(k-1)\tau\right)^k}{k!}, & " \end{cases} \tag{1}$$

Where t is time independent variable, τ is a constant delay term, B is $N \times N$ matrix, $\varnothing$ and I are respectively zero and identity matrices. The purpose of this matrix construction is to derive the representation of solutions of linear DDEs with a single constant delay. Their result can be recalled as follows:

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Theorem 1.1 [1] Suppose $\tau > 0$ and let B be an $N \times N$ matrix, $\phi \in C^1([-\tau, 0], \mathbb{R}^N)$ and $f : [0, \infty) \to \mathbb{R}^N$ be a given function. Then the solution of the Cauchy problem consisting of the equation

$$y'(t) = By(t-\tau) + f(t), t \geq 0,$$

with initial condition

$$y(t) = \vartheta(t), t \in [-\tau, 0], \tag{2}$$

has the form,

$$y(t) = e_\tau^{Bt}(-\tau) + \int_{-\tau}^{0} e_\tau^{B(t-\tau-s)} \vartheta'(s)\, ds + \int_0^t e_\tau^{B(t-\tau-s)} f(s)\, ds,$$

for any $t \geq -\tau$ and e_τ^{Bt} is defined in Equation (1). The generalisation of this result for $n \in \mathbb{N}$ constants delays was given Michal in 2012. This result can also be recalled as

Theorem 1.2 [2] Let $0 < \tau_1, \tau_2, \ldots, \tau_n \in \mathbb{R}$ for $n \in \mathbb{N}$, such that $\tau := max[\tau_1, \ldots, \tau_n]$, $B_1, B_2 \ldots, B_N$ *are* $N \times N$ pairwise permutable matrices, $\phi \in C([-\tau, 0], \mathbb{R})$, and $f : [0, \infty) \to \mathbb{R}^N$ be a given function. Then the solution of the Cauchy problem consisting of the equation

$$y'(t) = B_1 y(t-\tau_1) + B_2 y(t-\tau_2) + \ldots + B_n y(t-\tau_n) + f(t) \tag{3}$$

with initial condition in Equation (2) has the form

$$y(t) = \begin{cases} \vartheta, & -\tau \leq t < 0 \\ Y_n \vartheta(0) + \int_0^t Y_n(t-s) \sum_{m=1}^{n} B_m \psi(s-\tau_n)\, ds + & t \geq 0 \\ \int_0^t Y_n(t-s) f(t)\, ds, & '' \end{cases} \tag{4}$$

where

$$\psi = \begin{cases} \vartheta(t) & t \in [-\tau, 0) \\ \theta & t \notin [-\tau, 0) \end{cases}. \tag{5}$$

θ is N−dimentional vector of zeros and $Y_n(t) = e_{\tau_1, \tau_2, \ldots, \tau_n}^{B_1, B_2, \ldots, B_n(t-\tau)}$ is the multiple delay matrix exponential define as

$$e_{\tau_1, \tau_2, \ldots, \tau_n}^{B_1, B_2, \ldots, B_n t} = \begin{cases} \varnothing, & t \leq \tau_j \\ Y_{j-1}(t+\tau) & -\tau_j \leq t < 0 \\ Y_{j-1}(t+\tau_j) + & (k-1)\tau_j \leq t < k\tau_j, k \in N \\ B_j \int_0^t Y_{j-1}(t-s_1) Y_{j-1}(s_1)\, ds + & '' \\ \ldots + B_j^k \int_{(k-1)\tau_j}^{t} \int_{(k-1)\tau_j}^{s_1} \ldots & '' \\ \int_{(k-1)\tau_j}^{s_{k-1}} Y_{j-1}(t-s_1) \times & '' \\ \prod_{i=1}^{k-1} Y_{j-1}(s_i - s_{i+1}) \times & '' \\ Y_{j-1}(s_i - s_{i+1})\, ds_k \ldots ds_1 & '' \end{cases} \tag{6}$$

For $j = 2,3, .., n$ and $Y_{j-1}(t) = e_{\tau_1, \tau_2, \ldots, \tau_n}^{B_1, B_2, \ldots, B_{j-1}(t-\tau_j - 1)}$. Michal and Frantisek used unilateral Laplace transform and derived the representation of solutions for linear nonhomogeneous differential equations with any finite number of constants delays and pairwise permutable $N \times N$ matrices which another extension of Khusainove and Shuklin work. They later consider the following equation,

$$y'(t) = Ay(t) B_1 y(t-\tau_1) + B_2 y(t-\tau_2) + \ldots + B_n y(t-\tau_n) + f(t), \tag{7}$$

with Equation (2) as initial condition . To obtain the closed-form solution of Equation (7) Michal and Frantisek first transformed the equation to linear DDEs with multiple delays in which the delay independent term $Ay(t) = 0$. By this transformation, they used their derived result to obtain the representation of the solution of Equation (7). However, this result is not used to determine any solution properties of the corresponding

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nonlinear equation, but rather it was only meant for practical calculation of linear problems of the form of Equation (7).

Therefore, in this work, the Natural transform is applied to derive another representation of Equation (7). The idea for choosing Natural transform over Laplace transform here is because the Natural transform generalised both the theory of Laplace and Sumudu transforms, and also the application of Natural Transform to delay differential equations is very limited. In addition, our derived result is not only meant for practical calculation, but it can be also used to study the exponential stability of solution for the following nonlinear systems of DDEs.

$$y^{'}(t) = Ay(t)B_1y(t-\tau_1)+...+B_ny(t-\tau_n)+ F\big(y(t), y(t-\tau_1),...,y(t-\tau_n)\big), \tag{8}$$

with Equation (2) as initial condition. The sufficient conditions for the exponential stability of the trivial solution of Equation (8) are derived using the stability criteria provide by Medved and Michal in 2012. In this work they consider different form of nonlinearity based on the following definition.

Defifinition 1.1 Let $B_1, B_2, \ldots, B_n$ be $N \times N$ matrices and $f : [0, \infty)$ be a given function. Suppose $0 < \tau_1, \tau_2, \ldots, \tau_n \in \mathbb{R}$ such that $\tau = max\{\tau_1, \tau_2, \ldots, \tau_n\}$ for $n \in \mathbb{N}$ and $\phi \in ([-\tau, 0], \mathbb{R})$. If a function $y(t) \in C([-\tau, 0), \mathbb{R}) \cap C^1([0, \infty), \mathbb{R})$ (Only the right-hand derivative to be considered at $t = 0$) solves Equation (3) and also satisfifies the initial condition in Equation (2), then y(t) is a solution of the Cauchy problem defined in Equations (2) and (3).

MATERIALS AND METHODS

The definition and some important properties of Natural transform for further use in this research are rendered here. First, recall the defifinition of Heaviside unit step function $H(t)$ given by

$$H(t) = \begin{cases} 1, & t \geq 1 \\ 0, & t < 0 \end{cases},$$

$$H_\tau(t) = H(t-\tau) = \begin{cases} 1, & t \geq \tau \\ 0, & t < \tau \end{cases}$$

For $0 \leq t \in \mathbb{R}$, then the Natural transform (N^+) of the function $y(t)H(t) = y(t)$ defined on $\mathbb{R}^+$ can be defined over a set

$$A = \left\{ y(t) : \exists M \tau_1, \tau_2 \geq 0 \left| y(t) \right| < Me^{\frac{|t|}{\tau_j}}, t \in (-1)^j \times [0,\infty), j \in Z^+ \right\}$$

As in the given integral:

$$N^+[y(t)] = Y^+(s,u) = \int_0^\infty e^{-st} y(ut)dt; s,u \in [0,\infty). \tag{9}$$

Lemma 2.1 Let $F(s, u)$ and $G(s, u)$ be the Natural transform of the functions $f(t)$, $g(t) > 0$. Then for $t \geq 0$, the following properties of Natural transform hold (see Fethi *el al.*)

$$1. N^+[af(t)+bg(t)] = aN^+[f(t)]+bN^+[g(t)], a,b \in R$$

$$2. N^+[f^{'}(t)] = \frac{s}{u}N^+[f(t)] - \frac{f(0)}{u},$$

$$3. N^-\{uF(s,u)G(s,u)\} = (f*g)t,$$

$$4. N^-\left\{\frac{e^{-\frac{st}{u}}}{s}\right\} = H(t-\tau), \tau > 0,$$

$$5. N^-\{su^{n-1}H_1(s,u)H_2(s,u)...H_n(s,u)\} = (h_1*h_2*...*h_n), n \geq 3,$$

where $*$ is a convolution operator, N^- denote the inverse of N^+, $H_1(s,u),...,H_n(s,u)$ are respectively the Natural transform of and $h_1, h_2, ..., h_n > 0$ such that

$$(h_1*h_2*...*h_n) = \int_0^t h_1(a)h_2(a)...h_n(t-a)da.$$

Definition2.1 Let $\alpha, \tau \in \mathbb{R}$ such that $\alpha, \tau \geq 1$ and $H(t)$ be Heaviside unit step function, then for simplicity define a function

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$$I_{\alpha\tau}^{H,t} = \frac{(t-\alpha\tau)^{\alpha-1}}{(\alpha-1)!} H(t-\alpha\tau), t \geq 0.$$

Lemma 2.2 Let $\alpha \in \mathbb{N}$ and $\tau > 0$ then

$$N^{-}\left\{ s^{\chi_\alpha} u^{\gamma_\alpha(\alpha-1)} \left(\frac{e^{\frac{-s\tau}{u}}}{s} \right)^{\alpha} \right\} = I_{\alpha\tau}^{H,t}, t \geq 0,$$

where

$$\chi_\alpha = \begin{cases} 1, & \alpha \geq 3 \\ 0 & \alpha < 3 \end{cases} \text{ and } \gamma_\alpha = \begin{cases} 1, & \alpha \geq 2 \\ 0 & \alpha < 2 \end{cases}$$

Proof

Using an induction method, for $\alpha = 1$ the result follows from number 4 of Lemma 2.1. Suppose the statement is true for $\alpha = k$, then it is suffices to show the statement also holds for $\alpha = k + 1$. Now, from number 3 of Lemma 2.1 we have

$$N^{-}\left\{ su^{k} \left(\frac{e^{\frac{-s\tau}{u}}}{s} \right)^{k+1} \right\}$$

$$= N^{-}\left\{ u \left[su^{k-1} \left(\frac{e^{\frac{-s\tau}{u}}}{s} \right)^{k} * \left(\frac{e^{\frac{-s\tau}{u}}}{s} \right) \right] \right\}$$

$$= \int_0^t \frac{(a-k\tau)^{k-1}}{(k-1)!} H(a-k\tau) H(t-a-\tau) da$$

$$= \int_{k\tau}^{t-\tau} \frac{(a-k\tau)^{k-1}}{(k-1)!} H(t-(k+1)\tau) da$$

$$= \frac{(t-(k+1)\tau)^{k}}{k!} H(t-(k+1)\tau) = I_{(k+1)\tau}^{H.t}$$

This completes the proof of the Lemma. Therefore, based on the Definition 2.1 the inverse Natural transform of n multiple of Heaviside functions was successfully established. The generalization of this result is given in the following Lemma 2.3.

Lemma 2.3 Let $\alpha_i \in \mathbb{N}$ and $\tau > 0$ then

$$N^{-}\left\{ s^{\chi_n} u^{\gamma_n(n-1)} \prod_{i=1}^{n} s^{n} u^{\alpha_i - 1} \left(\frac{e^{\frac{-s\tau}{u}}}{s} \right)^{\alpha_i} \right\} \qquad (10)$$

$$= I_{\alpha_1\tau_1, \alpha_2\tau_2, \ldots, \alpha_n\tau_n}^{H,t}, n \in N,$$

where

$$\chi_n = \begin{cases} 1, & \alpha \geq 3 \\ 0 & \alpha < 3 \end{cases} \text{ and } \gamma_n = \begin{cases} 1, & \alpha \geq 2 \\ 0 & \alpha < 2 \end{cases}$$

Proof

Also by induction method with respect to n. Now, for $n = 1$ the result is obtained from Lemma 2.2. Suppose the result is also true for $n = p$. Let left-hand side of Equation (10) be denoted by P_n then

$$P_{p+1}=\left(P_p * N^{-}\left\{\left(\frac{e^{\frac{-st}{u}}}{s}\right)^{\alpha_p+1}\right\}\right)(t)$$

$$=\int_0^t \frac{\left(a-\sum_{i=1}^{p}\alpha_i\tau_i\right)^{\sum_{i=1}^{p}\alpha_i-1}}{\left(\sum_{i=1}^{p}\alpha_i-1\right)!}\times H\left(a-\sum_{i=1}^{p}\alpha_i\tau_i\right)\frac{\left(t-a-\alpha_{p+1}\tau_{p+1}\right)^{\alpha_{p+1}-1}}{\left(\alpha_{p+1}-1\right)!}\times H\left(t-a-\alpha_{p+1}\tau_{p+1}\right)da$$

$$=\int_{\sum_{i=1}^{p}\alpha_p\tau_i}^{t-\alpha_{p+1}\tau_{p+1}} \frac{\left(a-\sum_{i=1}^{p}\alpha_i\tau_i\right)^{\sum_{i=1}^{p}\alpha_i-1}}{\left(\sum_{i=1}^{p}\alpha_i-1\right)!}\times\frac{\left(t-a-\alpha_{p+1}\tau_{p+1}\right)^{\alpha_{p+1}-1}}{\left(\alpha_{p+1}-1\right)!}H\left(t-\sum_{i=1}^{p+1}\alpha_i\tau_i\right)da.$$

Now, the substitution of

$$a=\sum_{i=1}^{p}\alpha_i\tau_i+\xi\left(t-\sum_{i=1}^{p+1}\alpha_i\tau_i\right)$$

Yields

$$P_{p+1}=\frac{\left(t-\sum_{i=1}^{p+1}\alpha_i\tau_i\right)^{\sum_{i=1}^{p+1}\alpha_i-1}H\left(t-\sum_{i=1}^{p+1}\alpha_i\tau_i\right)}{\left(\sum_{i=1}^{p}\alpha_i-1\right)!\left(\alpha_{p+1}-1\right)!}B\left(\sum_{i=1}^{p}\alpha_i,\alpha_{p+1}\right),$$

where $B(.,\ .)$ is the Eula beta function. By rewritting this function using gamma functions $B(u,v)=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$ this leads to

$$P_{p+1}=\frac{\left(t-\sum_{i=1}^{p+1}\alpha_i\tau_i\right)^{\sum_{i=1}^{p+1}\alpha_i-1}H\left(t-\sum_{i=1}^{p+1}\alpha_i\tau_i\right)}{\left(\sum_{i=1}^{p}\alpha_i-1\right)!}=I^{H,t}_{\alpha_1\tau_1,\alpha_2\tau_2,\ldots,\alpha_{p+1}\tau_{p+1}}.$$

The the proof is now completed.

Lemma 2.4 Suppose the assumptions of Lemma 2.3 are satisfied. If $B_1, B_2, \ldots, B_n$ are $N\times N$ matrices and ω is an N−dimentional vector then

$$N^{-}\left\{s^{\chi_n}u^{\gamma_n(n-1)}\prod_{i=1}^{n}s^n u^{\alpha_{i-1}}\left(\frac{B_i e^{\frac{-s\tau}{u}}}{s}\right)^{\alpha_i}\omega\right\}$$

$$=\left(\prod_{i=1}^{n}B_i^{\alpha_i}\right)\omega I^{H,t}_{\alpha_1\tau_1,\alpha_2\tau_2,\ldots,\alpha_n\tau_n},n\in N. \tag{11}$$

Proof

Consider the following equation

$$N^{-}\left\{s^{\chi_n}u^{\gamma_n(n-1)}\prod_{i=1}^{n}s^n u^{\alpha_{i-1}}\left(\frac{B_i e^{\frac{-s\tau}{u}}}{s}\right)^{\alpha_i}\omega\right\}$$

$$=N^{-}\left\{s^{\chi_n}u^{\gamma_n(n-1)}\prod_{i=1}^{n}s^n u^{\alpha_{i-1}}\left(\frac{e^{\frac{-s\tau}{u}}}{s}\right)^{\alpha_i}\right\}\left(\prod_{i=1}^{n}B_i^{\alpha_i}\right)\omega$$

$$=\left(\prod_{i=1}^{n}B_i^{\alpha_i}\right)\omega I^{H,t}_{\alpha_1\tau_1,\alpha_2\tau_2,\ldots,\alpha_n\tau_n}.$$

The proof is now completed.

2.1 Derivation of closed-form formula

To obtain the main result of this work it is necessary to get the sufficient condition for the solution $y(t)$ of Equation (3) to be an element of set A. In such case, there is need to recall the result of multi-delayed matrix exponential $e^{B_1,B_2,\ldots B_n(t-\tau_n)}_{\tau_1,\tau_2,\ldots\tau_n}$ obtained in Medved and Michal work as given in the following Lemma 2.5.

Lemma 2.5 [4] Let $n>0, \tau_1,\tau_2,\ldots,\tau_n \epsilon \mathbb{R}$, and $B_1, B_2, \ldots n$ be as defined in Theorem 1.2. If $\|B_i\|\le\alpha_i e^{\alpha_i\tau_i}$ for each $i=1,2,\ldots,n$ then

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$$\left\| e^{B_1, B_2, \ldots B_n (t-\tau_n)}_{\tau_1, \tau_2, \ldots \tau_n} \right\| \le e^{(\alpha_1+\alpha_2+\ldots+\alpha_n)t}, t \in R.$$

Lemma 2.6 Suppose all the assumptions of Theorem 1.2 are satisfied and let the function f be an element of set A. Then the solution $y(t)$ of Equations (2) and (3) is also in A.

Proof

Let $0 < M, c_1 \epsilon \mathbb{R}$ such that $|f(t)| \le M e^{\frac{|t|}{c_1}}$ for every $t \in (-1)^j \times [0, \infty), j \in \mathbb{Z}^+$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that $\|B_i\| \le \alpha_i e^{\alpha_i \tau_i}$ then from Lemma 2.5, $\|X_n(t)\| \le e^{\alpha t}$ for any $t \in \mathbb{R}$ for $\alpha = \sum_{i=1}^{n} \alpha_i$. Let $\Omega = \max_{[-\tau, 0]} |\varphi(t)|$ for $t \ge 0$, then we have

$$\begin{aligned} |y(t)| &\le \Omega e^{\alpha t} + \sum_{i=1}^{n} \Omega \|B_i\| \\ &\times \int_0^t e^{\alpha(t-s)} ds + M \int_0^t e^{\alpha(t-s)+\frac{|s|}{c_1}} ds \\ &\le \Omega \left(1 + \sum_{i=1}^{n} \frac{\|B_i\|}{\alpha} \right) e^{\alpha t} \\ &+ \frac{M c_1}{\alpha} e^{\alpha\left(\alpha+\frac{1}{c_1}\right)|t|}. \end{aligned} \tag{12}$$

For a very large value of c_1 then

$$|y(t)| \le \Omega \left(1 + \sum_{i=1}^{n} \frac{\|B_i\|}{\alpha} \right) e^{\alpha t}$$

$$\frac{M c_1}{\alpha} e^{\alpha\left(\alpha+\frac{1}{c_1}\right)|t|} \le k e^{\frac{1}{c}|t|}.$$

where $K, c \ge (\alpha + c_1)$ are constants. Hence $y(t)$ is in set A.

So, Theorem 2.1 below gave the closed-form formula for the representation of the solution of Equations (2) and (3) and proved using the essential notions of NT.

Theorem 2.1 Let $B_1, B_2, \ldots, B_n$ be pairwise permutable $N \times N$ that is, $B_i B_j = B_j B_i$ for every $i, j = 1, 2, \ldots, n \in \mathbb{N}$ and $\tau_1, \tau_2, \ldots, \tau_n > 0$ such that $\tau = \{\tau_1, \tau_2 \ldots, \tau_n\}$. If $\varphi \in ([0, \tau], \mathbb{R}^N)$ and f belongs to set A (there exist some constants $M, c_1 > 0$ such that $|f(t)| < M e^{\frac{|t|}{c_1}}$ for all $t \in (-1)^j \times [0, \infty), j \in \mathbb{Z}^+$ Then the solution of the Cauchy problem consists of Equations (2) and (3) can be expressed in the following form

$$y(t) = \begin{cases} \varphi(t), & -\tau \le t < 0 \\ X(t)\varphi(0) + \sum_{n=1}^{n} B_i \int_0^{\tau_i} X(t-a) & t \ge 0 \\ \times \varphi(a-\tau_i) da + \int_0^t X(t-a) f(a) da & " , \end{cases} \tag{13}$$

where

$$X(t) = \sum_{\sum_{i=1}^{n} \alpha_i \tau_i \le t} \begin{pmatrix} \alpha \\ \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix} \prod_{i=1}^{n} B_i^{\alpha_i} \times \sum_{m=0}^{n} \frac{(-1)^m t^{n-m} \left(\sum_{n=1}^{n} \alpha_i \tau_i \right)^{\alpha+m}}{(n-m)!(\alpha+m)!}, \tag{14}$$

for $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$,

$$\begin{pmatrix} \alpha \\ \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix} = \frac{\alpha}{\alpha_1! \alpha_2! \ldots \alpha_n!}.$$

Proof

By Lemma 2.6 the solution, $y(t) \in A$ therefore, the Natural transform can be applied to study Equation (3). Hence, from 2 of Lemma 2.1 the following are obtained

$$\begin{aligned} \frac{u}{s} N^+[y(t)] &= \frac{\varphi(0)}{u} + \sum_{i=1}^{n} B_i \int_0^t e^{-st} y(ut-\tau_i) ds + N^+[f(t)] \\ &= \frac{\varphi(0)}{u} + \sum_{i=1}^{n} B_i \left(\int_0^{\tau_i} e^{-st} \varphi(ut-\tau_i) dt + \int_{\tau_i}^{\infty} e^{-st} y(ut-\tau_i) dt \right) + F(,) \\ &= \frac{\varphi(0)}{u} + \sum_{i=1}^{n} B_i \left(\int_0^{\infty} e^{-st} \psi(ut-\tau_i) dt + e^{\frac{-s\tau_i}{u}} \int_0^{\infty} e^{-st} y(ut) dt \right) + F(,) \\ &= \frac{\varphi(0)}{u} + \sum_{i=1}^{n} \left(B_i N^+[\psi(t-\tau_i)] + B_i e^{\frac{-s\tau_i}{u}} N^+[y(t)] + F(s,u) \right), \end{aligned}$$

$$\Rightarrow \left(\frac{s}{u} I - \sum_{i=1}^{n} B_i e^{\frac{-s\tau_i}{u}} \right) N^+[y(t)] = \frac{\varphi(0)}{u} + \sum_{i}^{n} B_i N^+[\psi(t-\tau)] + F(,)$$

where ψ is defined in Equation (5). Suppose

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that, the fraction $\frac{u}{s}$ is large enough such that $\left\|\sum_{i=1}^{n} B_i\, e^{\frac{-s\tau_i}{u}}\right\| \leq \frac{s}{u}$, where $\|.\|$ is a fixed induced norm. Then by the theory of matrices $I - \sum_{i=1}^{n} \frac{\sum_{i=1}^{n} uB_i\, e^{\frac{-s\tau_i}{u}}}{s}$ is non-singular. Hence the following holds

$$\left(I - \sum_{i=1}^{n} \frac{uB_i e^{\frac{-s\tau_i}{u}}}{s}\right)^{-1} = \sum_{\alpha=0}^{\infty}\left(\sum_{i=1}^{n} \frac{uB_i e^{\frac{-s\tau_i}{u}}}{s}\right)^{\alpha}$$

$$\Rightarrow N^{+}\left[y(t)\right] = \left(I - \sum_{i=1}^{n} \frac{uB_i e^{\frac{-s\tau_i}{u}}}{s}\right)^{-1}$$

$$\times\left[\frac{u}{s}\left(\frac{\varphi(0)}{u} + \sum_{i}^{n} B_i N^{+}\left[\psi(t-\tau_i) + F(s,u)\right]\right)\right]$$

Therefore, the solution $y(t)$ can be expressed as follows

$$y(t) = X_0 + \sum_{j=1}^{n} B_j X_j + X_f,$$

where

$$X_0 = N^{-}\left\{\frac{u}{s}\left(\sum_{\alpha=0}^{\infty}\left(\sum_{i=1}^{n} \frac{uB_i e^{\frac{-s\tau_i}{u}}}{s}\right)^{\alpha}\right)\frac{\varphi(0)}{u}\right\}$$

$$X_j = N^{-}\left\{\frac{u}{s}\left(\sum_{\alpha=0}^{\infty}\left(\sum_{i=1}^{n} \frac{uB_i e^{\frac{-s\tau_i}{u}}}{s}\right)^{\alpha}\right)N^{+}\left[\psi(t-\tau_j)\right]\right\}$$

$$X_f = N^{-}\left\{\frac{u}{s}\left(\sum_{\alpha=0}^{\infty}\left(\sum_{i=1}^{n} \frac{uB_i e^{\frac{-s\tau_i}{u}}}{s}\right)^{\alpha}\right)F(s,u)\right\}.$$

So, by using multinomial Theorem we obtain the following equation

$$X_0 = N^{-}\left\{\frac{u}{s}\left(\sum_{\alpha=0}^{\infty}\left(\sum_{i=1}^{n} \frac{uB_i e^{\frac{-s\tau_i}{u}}}{s}\right)^{\alpha}\right)\frac{\varphi(0)}{u}\right\}$$

$$= \sum_{\alpha=1}^{\infty} \sum_{\substack{|\alpha|=\alpha \\ \alpha_1,\alpha_2,\ldots,\alpha_n \geq 0}} \binom{\alpha}{\alpha_1,\alpha_2,\ldots,\alpha_n} N^{-}\left\{\frac{1}{s}\left(\prod_{i=1}^{n} \frac{uB_i e^{\frac{-s\tau_i}{u}}}{s}\right)^{\alpha_i} \varphi(0)\right\}$$

$$= \sum_{\alpha=1}^{\infty} \sum_{\substack{|\alpha|=\alpha \\ \alpha_1,\alpha_2,\ldots,\alpha_n \geq 0}} \binom{\alpha}{\alpha_1,\alpha_2,\ldots,\alpha_n} N^{-}\left\{\frac{u^{n+2}}{s^{n+1}} s^{\chi_\alpha} u^{\gamma_\alpha(n-1)}\right.$$

$$\left.\times\left(\prod_{i=1}^{n} B_i s^n u^{\alpha_i - 1} \frac{e^{\frac{-s\tau_i}{u}}}{s}\right)^{\alpha_i} \varphi(0)\right\}$$

$$= \sum_{\alpha=1}^{\infty} \sum_{\substack{|\alpha|=\alpha \\ \alpha_1,\alpha_2,\ldots,\alpha_n \geq 0}} \binom{\alpha}{\alpha_1,\alpha_2,\ldots,\alpha_n} N^{-}\left\{u^2 s\left[\frac{1}{s}\frac{u^n}{s^{n+1}} s^{\chi_\alpha} u^{\gamma_\alpha(n-1)}\right]\right.$$

$$\left.\times\left(\prod_{i=1}^{n} B_i s^n u^{\alpha_i - 1} \frac{e^{\frac{-s\tau_i}{u}}}{s}\right)^{\alpha_i} \varphi(0)\right\}. \tag{15}$$

Now, apply 5 of Lemma 2.1 on Equation (15) to get

$$X_0 = \sum_{\alpha=1}^{\infty} \sum_{\substack{|\alpha|=\alpha \\ \alpha_1,\alpha_2,\ldots,\alpha_n \geq 0}} \binom{\alpha}{\alpha_1,\alpha_2,\ldots,\alpha_n}\left\{H(t) * \frac{t^n}{n!}\right.$$

$$\left. * N^{-}\left[s^{\chi_\alpha} u^{\gamma_\alpha(n-1)}\left(\prod_{i=1}^{n} B_i s^n u^{\alpha_i - 1} \frac{e^{\frac{-s\tau_i}{u}}}{s}\right)^{\alpha}\right]\varphi(0)\right\}. \tag{16}$$

Applying Lemma 2.4 on Equation (16) to have

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$$X_0 = \sum_{\alpha=1}^{\infty} \sum_{\substack{|\alpha|=\alpha \\ \alpha_1,\alpha_2,\ldots,\alpha_n \geq 0}} \binom{\alpha}{\alpha_1,\alpha_2,\ldots,\alpha_n} \int_0^t \frac{\left(a - \sum_{i=1}^{n} \alpha_i \tau_i\right)^{\sum_{i=1}^{n}\alpha_i - 1}}{\left(\sum_{i=1}^{n} \alpha_i - 1\right)!}$$

$$\times \left(\prod_{i=1}^{n} B_i^{\alpha_i}\right) \varphi(0) \frac{\alpha^n}{n!} H\left(a - \sum_{i=1}^{n} \alpha_i \tau_i\right) H(t-a)\, da$$

$$= \sum_{\alpha=1}^{\infty} \sum_{\substack{|\alpha|=\alpha \\ \alpha_1,\alpha_2,\ldots,\alpha_n \geq 0}} \binom{\alpha}{\alpha_1,\alpha_2,\ldots,\alpha_n} \left(\prod_{i=1}^{n} B_i^{\alpha_i}\right) \varphi(0) H\left(t - \sum_{i=1}^{n} \alpha_i \tau_i\right)$$

$$\times \frac{t^n \left(t - \sum_{i=1}^{n} \alpha_i \tau_i\right)^{\alpha}}{n!\alpha!} - \frac{t^{n-1} \left(t - \sum_{i=1}^{n} \alpha_i \tau_i\right)^{\alpha+1}}{(n-1)!(\alpha+1)!} + \frac{t^{n-2} \left(t - \sum_{i=1}^{n} \alpha_i \tau_i\right)^{\alpha+2}}{(n-2)!(\alpha+2)!}$$

$$\pm \ldots \pm \frac{t \left(t - \sum_{i=1}^{n} \alpha_i \tau_i\right)^{(\alpha+(n-1))}}{(\alpha+(n-1))!} \pm \frac{\left(t - \sum_{i=1}^{n} \alpha_i \tau_i\right)^{\alpha+n}}{(\alpha+n)!}$$

$$= \sum_{\alpha=1}^{\infty} \sum_{\substack{|\alpha|=\alpha \\ \alpha_1,\alpha_2,\ldots,\alpha_n \geq 0}} \binom{\alpha}{\alpha_1,\alpha_2,\ldots,\alpha_n} \left(\prod_{i=1}^{n} B_i^{\alpha_i}\right) \varphi(0) \tag{17}$$

$$\times \left(\sum_{m=0}^{n} \frac{(-1)^m t^{n-m} \left(t - \sum_{i=1}^{n} \alpha_i \tau_i\right)^{\alpha+m}}{(n-m)!(\alpha+m)!}\right) = X(t)\varphi(0).$$

Also, by using multinomial Theorem

$$X_j = N^- \left\{ \frac{u}{s} \left(\sum_{\alpha=0}^{\infty} \left(\sum_{i=1}^{n} \frac{u B_i e^{\frac{-s\tau_i}{u}}}{s} \right)^{\alpha} \right) N^+ \left[\psi(t - \tau_j) \right] \right\} \tag{18}$$

$$\sum_{\alpha=1}^{\infty} \sum_{\substack{|\alpha|=\alpha \\ \alpha_1,\alpha_2,\ldots,\alpha_n \geq 0}} \binom{\alpha}{\alpha_1,\alpha_2,\ldots,\alpha_n} N^- \left\{ u \left(\frac{1}{s} \prod_{i=1}^{n} \frac{u B_i e^{\frac{-s\tau_i}{u}}}{s} \right)^{\alpha_i} N^+ \left(\psi(t - \tau_j) \right) \right\}$$

Now by 3 of Lemma 2.1 Equation (18) leads to

$$X_j = \sum_{\alpha=1}^{\infty} \sum_{\substack{|\alpha|=\alpha \\ \alpha_1,\alpha_2,\ldots,\alpha_n \geq 0}} \binom{\alpha}{\alpha_1,\alpha_2,\ldots,\alpha_n} \left[\psi(t - \tau_j) \right] *$$

$$N^- \left\{ u^2 s \left[\frac{1}{s} \frac{u^n}{s^{n+1}} s^{\chi_n} u^{\gamma_n(n-1)} \left(\prod_{i=1}^{n} B_i s^n u^{\alpha_i - 1} \frac{e^{\frac{-s\tau_i}{u}}}{s} \right)^{\alpha_i} \right] \right\}. \tag{19}$$

Now, apply 5 of Lemma 2.1 on the left-hand side of Equation (19) to get

$$X_j = \sum_{\alpha=1}^{\infty} \sum_{\substack{|\alpha|=\alpha \\ \alpha_1,\alpha_2,\ldots,\alpha_n \geq 0}} \binom{\alpha}{\alpha_1,\alpha_2,\ldots,\alpha_n} \left[\psi(t - \tau_j) \right] *$$

$$\left\{ H(t) * \frac{t^n}{n!} N^- \left[s^{\chi_n} u^{\gamma_n(n-1)} \left(\prod_{i=1}^{n} B_i s^n u^{\alpha_i - 1} \frac{e^{\frac{-s\tau_i}{u}}}{s} \right)^{\alpha_i} \right] \right\}. \tag{20}$$

Therefore, using Equation (17) and 5 of Lemma 2.1 Equation (20) becomes

$$X_j = \int_0^t \sum_{\substack{\sum_{i=1}^{n} \alpha_i \tau_i \leq t \\ \alpha_1,\alpha_2,\ldots,\alpha_n}} \binom{\alpha}{\alpha_1,\alpha_2,\ldots,\alpha_n} \left(\prod_{i=1}^{n} B_i^{\alpha_i}\right) \times \tag{21}$$

$$\left(\sum_{m=0}^{n} \frac{(-1)^m (t-a)^{n-m} \left(t - a - \sum_{i=1}^{n} \alpha_i \tau_i\right)^{\alpha+m}}{(n-m)!(\alpha+m)!} \right) \psi(t - \tau_j)\, da.$$

Now, for every $j = 1,2, \ldots, n$ the integral in Equation (21) can be compressed to $\int_0^{\tau_j}$ .when $\tau_j < t$ and lead to $\psi(a - \tau_j) \to \phi(a - \tau_j)$. And from the other hand, i.e. for $\tau_j > t$, this can be extended to $\int_0^{\tau_j}$.

Hence,

$$X_j = \int_0^{\tau_j} X(t-a) \varphi(a - \tau_j)\, da, \; j = 1,2,\ldots,n.$$

Lastly, by replecing $N^+[\psi(t - \tau_j)]$ with $F(s,u)$ in Equation (18) and following the same process then X_f can be derived as

$$X_j = \int_0^t \sum_{\substack{\sum_{i=1}^{n} \alpha_i \tau_i \leq t \\ \alpha_1,\alpha_2,\ldots,\alpha_n}} \binom{\alpha}{\alpha_1,\alpha_2,\ldots,\alpha_n} \left(\prod_{i=1}^{n} B_i^{\alpha_i}\right) \times$$

$$\left(\sum_{m=0}^{n} \frac{(-1)^m (t-a)^{n-m} \left(t - a - \sum_{i=1}^{n} \alpha_i \tau_i\right)^{\alpha+m}}{(n-m)!(\alpha+m)!} \right) f(a)\, da \tag{22}$$

$$= \int_0^t X(t-a) f(a)\, da.$$

And this completes the proof.

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Corollary 2.1 Suppose $f, \varphi, \tau_1, \tau_2, ...\tau_n$, and τ be define in the manner as in Theorem 3.1. Let $A, B_1, B_2, ..., B_n$ be pairwise permutable $N \times N$ matrices, then the solution of the Equations (7) and (2) has the form

$$y(t)=\begin{cases}\varphi(t), & -\tau\le t<0\\ X_A(t)\varphi(0)+\sum_{n=1}^{n}B_i\int_0^{\tau_i}X_A(t-a) & t\ge 0\\ \times\varphi(a-\tau_i)da+\int_0^t X_A(t-a)f(a)da, & "\end{cases} \tag{23}$$

where

$$X_A(t)=e^{At}\sum_{\substack{\sum_{i=1}^{n}\alpha_i\tau_i\le t\\ \alpha_1,\alpha_2,...\alpha_n}}\binom{\alpha}{\alpha_1,\alpha_2,...,\alpha_n}\times\left(\sum_{m=0}^{n}\frac{(-1)^m t^{n-m}\left(t-\sum_{i=1}^{n}\alpha_i\tau_i\right)^{\alpha+m}}{(n-m)!(\alpha+m)!}\right)\prod_{i=1}^{n}\tilde{B}_i^{\alpha_i}, \tag{24}$$

for all $t \in \mathbb{R}$ such that $\tilde{B}_i = B_i e^{-A\tau_i}, i = 1,2,...n$.

Proof

By setting $y(t) = e^{At}v(t)$ then a new problem is obtained in v given by

$$v'(t)=\tilde{B}_1 v(t-\tau_1)+\tilde{B}_2 v(t-\tau_2)+....$$
$$+\tilde{B}_n v(t-\tau_n)+\tilde{f}(t), t\ge 0 \tag{25}$$
$$v(t)=e^{-At}\varphi(t)=\varphi(t), -\tau<t\le 0,$$

where $\tilde{f}(t)=e^{-At}$ for every $t \ge 0$. By Theorem 2.1 the solution to Equation (25) can be given as

$$v(t)=\begin{cases}\tilde{\varphi}(t), & -\tau\le t<0\\ \tilde{X}(t)\tilde{\varphi}(0)+\sum_{n=1}^{n}\tilde{B}_i\int_0^{\tau_i}\tilde{X}(t-a) & t\ge 0\\ \times\tilde{\varphi}(a-\tau_i)da+\int_0^t \tilde{X}(t-a)\tilde{f}(a)da, & "\end{cases} \tag{26}$$

Where

$$\tilde{X}(t)=e^{At}\sum_{\substack{\sum_{i=1}^{n}\alpha_i\tau_i\le t\\ \alpha_1,\alpha_2,...\alpha_n}}\binom{\alpha}{\alpha_1,\alpha_2,...,\alpha_n}\times\left(\sum_{m=0}^{n}\frac{(-1)^m t^{n-m}\left(t-\sum_{i=1}^{n}\alpha_i\tau_i\right)^{\alpha+m}}{(n-m)!(\alpha+m)!}\right)\prod_{i=1}^{n}\tilde{B}_i^{\alpha_i},$$

Now, by comming back to y and using $\tilde{\varphi}(0)=\varphi(0)$ implies that

$$e^{At}=\tilde{B}_i\tilde{X}(t-a)\tilde{\varphi}(t-\tau_j)=\tilde{B}_i e^{A(t-a)}\tilde{X}(t-a)\tilde{\varphi}(t-\tau_j)$$
$$e^{-At}\tilde{X}(t-a)\tilde{f}(a)=e^{At}\tilde{X}(t-a)f(a),$$

and $X_A(t)=e^{At}\tilde{X}(t)$. Hence the result follows immediately.

RESULTS

This section is designed to derive sufficient conditions that guarantee the exponential stability of the solution for a different form of nonlinear DDEs with multiple numbers of constant delays. Thus, it is important to recall the following definition.

Definition 3.1 Let $f: \mathbb{R}^N \times ... \times \mathbb{R}^N \to \mathbb{R}$ be a given function and $A, B_1, B_2, ..., B_n$ be pairwise $N \times N$ permutable matrices. Suppose $\tau_1, \tau_2, ..., \tau_n \ge 0$ with $\tau = max[\tau_1, \tau_2, ..., \tau_n]$ and $\varphi \in c_\tau^1$. Then the solution $x_\varphi[-\tau, \infty) \to \mathbb{R}^N$ of equation

$$y'(t)=Ay(t)B_1(t-\tau_1)+...+B_n(t-\tau_n)+ F\big(y(t),y(t-\tau_1),...,y(t-\tau_n)\big), \tag{27}$$

satisfies Equation (2) is said to be exponentially stable if there are some positive constants c_1, c_2 and δ which depend on $A, B_1, B_2, ..., B_n, f$ and $\|\varphi\|_1$ define by

$$\|\varphi\|_1 = \max_{[-\tau,0]} \|\varphi(t)\| + \max_{[-\tau,0]} \|\varphi'(t)\|$$

Such that

$$\|v_\psi - v_\varphi\| \leq c_1 e^{-c_2 t}, t \geq 0,$$

Where $v_\psi(t)$ is any solution of Equation (27) which satisfies the initial condition

$$v_\psi(t) = \psi(t), -\tau \leq t \leq 0$$

With $\psi \in c_\tau^1$ and $\|\psi(t) - \varphi(t)\| \leq \delta$.

Lemma 3.1 Let $n \in \mathbb{N}, 0 \leq \tau_1, \tau_2, ..., \tau_n$ and $B_1, B_2, ..., B_n$ b e pairwise permutable $N \times N$ matrices. If $\|B_i\| \leq \varepsilon_i e^{\varepsilon_i \tau_i}$ for $i = 1, 2, ..., n$, where $\varepsilon_i \in \mathbb{R}$ such that $\sum_{i=1}^{n} \varepsilon_i \tau_i \leq t$, then $|X(t)| \leq Ce^{\alpha t}$, for $0 \leq C \in \mathbb{R}$.

Proof

$$\|X(t)\| = \left\| \sum_{\substack{\sum_{i=1}^{n} \alpha_i \tau_i \leq t \\ \alpha_1, \alpha_2, ... \alpha_n \geq 0}} \left(\sum_{m=0}^{n} \frac{(-1)^m \alpha! t^{n-m} \left(t - \sum_{i=1}^{n} \alpha_i \tau_i\right)^\alpha \prod_{i=1}^{n} B_i^{\alpha_i}}{(n-m)!(\alpha+m)!\alpha_1!.\alpha_2!...\alpha_n!} \right) \right\| \tag{28}$$

$$\leq \left\| \sum_{\substack{\sum_{i=1}^{n} \alpha_i \tau_i \leq t \\ \alpha_1, \alpha_2, ... \alpha_n \geq 0}} \left(\sum_{m=0}^{n} \frac{(-1)^m \alpha! t^{n-m} \left(t - \sum_{i=1}^{n} \alpha_i \tau_i\right)^\alpha}{(n-m)!(\alpha+m)!\alpha_1!.\alpha_2!...\alpha_n!} \right) \right\| \|B_1^{\alpha_1}\|.\|B_2^{\alpha_2}\|...\|B_n^{\alpha_n}\|$$

$$\leq \left\| \sum_{\substack{\sum_{i=1}^{n} \alpha_i \tau_i \leq t \\ \alpha_1, \alpha_2, ... \alpha_n \geq 0}} \left(\sum_{m=0}^{n} \frac{(-1)^m \alpha! t^{n-m} \left(t - \sum_{i=1}^{n} \alpha_i \tau_i\right)^\alpha}{(n-m)!(\alpha+m)!\alpha_1!.\alpha_2!...\alpha_n!} \right) \right\| \left(\varepsilon_1 e^{\varepsilon_1 \tau_1}\right)^{\alpha_1} ....\left(\varepsilon_n e^{\varepsilon_n \tau_n}\right)^{\alpha_n}$$

Since *n, m* ≥ 0, therefore the following hold

$$\|X(t)\| \leq \left\| \sum_{\substack{\sum_{i=1}^{n} \alpha_i \tau_i \leq t \\ \alpha_1, \alpha_2, ... \alpha_n \geq 0}} \left(\sum_{m=0}^{n} \frac{t^n \left(t - \sum_{i=1}^{n} \alpha_i \tau_i\right)^\alpha}{(n-m)!\alpha_1!.\alpha_2!...\alpha_n!} \right) \right\| (\varepsilon_1.\varepsilon_2...\varepsilon_n)^\alpha e^{\alpha t} \tag{29}$$

$$\leq \left\| \sum_{\substack{\sum_{i=1}^{n} \alpha_i \tau_i \leq t \\ \alpha_1, \alpha_2, ... \alpha_n \geq 0}} \left(\sum_{m=0}^{n} \frac{t^n \left(\sum_{i=1}^{n} \alpha_i \tau_i - t^2\right)^\alpha}{(n-m)!\alpha_1!.\alpha_2!...\alpha_n!} \right) \right\| (\varepsilon_1.\varepsilon_2...\varepsilon_n)^\alpha e^{\alpha t}.$$

Since $\sum_{m=0}^{n} \frac{1}{(m-n)!}$ is convergent power series implies that

$$\|X(t)\| \leq \left\| \sum_{\substack{\sum_{i=1}^{n} \alpha_i \tau_i \leq t \\ \alpha_1, \alpha_2, ... \alpha_n \geq 0}} \left(\sum_{m=0}^{n} \frac{\left(\sum_{i=1}^{n} \alpha_i \tau_i - (\alpha_i \tau_i)^2\right)^\alpha}{(n-m)!\alpha_1!.\alpha_2!...\alpha_n!} \right) \right\| \leq Ce^{\alpha t}, \tag{30}$$

where $\tau = \max[\tau_i]$, a constant

$$C = \left| \frac{\left(\alpha\tau - (\alpha\tau)^2\right)^\alpha}{\alpha_1!\alpha_2!...\alpha_n!} \sum_{m=0}^{n} \frac{1}{(n-m)!} \right| (\varepsilon_1 \varepsilon_2 ... \varepsilon_n)^\alpha$$

and $\alpha = \sum_{i=1}^{n} \alpha_i$ such that $\alpha \geq t$.

Theorem 3.1

Let $A, B_1, B_2, ..., B_n$ be pairwise permutable $N \times N$ matrices, $\tau_1, \tau_2, ..., \tau_n \geq 0$ with $\tau = max[\tau_1, \tau_2, ..., \tau_n]$, where $n \geq 0$. Suppose the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of a matrix A posses the property that $Re\lambda_1 \leq Re\lambda_2 \leq ... \leq Re\lambda_n \leq -h \leq 0$. Let $\varepsilon_1, \varepsilon_1, ..., \varepsilon_1 \in \mathbb{R}$ such that $\|B_i\| \leq \varepsilon_i e^{\varepsilon_i \tau_i}$ for $i = 1,2, ..., n$. If $\varepsilon = \sum_{i=}^{n} \varepsilon_i$ and $f(y) = O(\|y\|)$ then the solution of the system

$$y'(t) = Ay(t)B_1(t-\tau_1) + ... + B_n(t-\tau_n) + f(y(t)) \tag{31}$$

is exponentially stable.

Proof

Since Equation (31) satisfies Equation (2) then according to Corollary 2.1 its solution has the following form

$$y(t) = X_A(t)\varphi(0) + \sum_{j=1}^{n} B_j \int_0^{\tau_i} X_A(t-a)\varphi(a-\tau_i)da$$

$$+ \int_0^t X_A(t-a) f(y(a))da,$$

where $X_A(t) = e^{At}X(t)$. From the given property of eigenvalues of A implies that there are some positive constants h, H such that $\|e^{At}\| \leq He^{-ht}$ for all $t \geq 0$. Also since $f(y) = O(\|y(y)\|)$ therefore for all $P > 0$ there exists a positive δ such that if $\|y\| < \delta$ implies that $\|f(y)\| > P\|y\|$. Hence, for $t \geq 0$ and $\|y(a)\|$ is sufficiently small for $a \in$

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$[0, t]$ then by applying these two estimations together with Lemma 3.1 yield

$$\|y(t)\| \leq Ke^{(\alpha-h)t}\varphi(0) + K\sum_{j=1}^{n}\|B_j\|\int_0^{\tau_i} e^{(\alpha-h)(t-a)} \times\|\varphi(a-\tau_j)\|da + K\int_0^t e^{(\alpha-h)(t-a)}\|y(a)\|da, \quad (32)$$

where $k = HC$. Assume that $\|y(a)\| \leq \delta$, for $a \in [0, t]$ and $t \geq 0$ and let $u(t) = e^{(\alpha-h)t}\|y(a)\|$ then define a function

$$M(\varphi, \varphi') = K\varphi(0) + K\sum_{j=1}^{n}\|B_j\|\int_0^{\tau_i} e^{(h-\alpha)a}\|\varphi(a-\tau_j)\| da. \quad (33)$$

Hence, the following inequality follows from Equations (32) and (33).

$$u(t) \leq M(\varphi, \varphi') + \int_0^t u(a)da. \quad (34)$$

By applying Gronwalls inequality, we obtained an estimated solution of Equation (31) as

$$\|y(t)\| = e^{(\alpha-k)t}u(t) \leq M(\varphi, \varphi')e^{(KP+\alpha-h)t}. \quad (35)$$

Now, for $t \geq 0$, if $\|\varphi\|_1$ 1 is sufficiently small and $P \leq \frac{h-\alpha}{k}$ then $0 \leq M(\varphi, \varphi')e^{-\gamma t}$ for $\gamma = h - \alpha - KP > 0$. Therefore, the solution of Equation (31) is exponentially stable.

Theorem 3.2 Suppose the assumptions of Theorem 3.1 are satisfied. Let $f(y, v_1, \ldots v_n) = O(\|y\| + \|v_1\| + \ldots + \|v_1\|)$ that is for all $P > 0$ there exists $\delta > 0$ such that if $\|y\|, \|v_1\|, \ldots, \|v_1\| < \delta$ implies $\|f(y, v_1, \ldots v_n)\| < P(\|y\| + \|v_1\| + \ldots + \|v_1\|)$. Then the solution of Equation (27) is exponentially stable.

Proof

Also, Equation (27) satisfies Equation (2) then according to Corollary 2.1 its solution has the form

$$y(t) = X_A(t)\varphi(0) + \sum_{j=1}^{n} B_j\int_0^{\tau_i} X_A(t-a)\varphi(a-\tau_i)\, da + \int_0^t X_A(t-a) f(y, v_1, v_2, \ldots, v_n)\, da,$$

From the given property of eigenvalues of A implies that there are some positive constants $h, H > 0$ such that $\|e^{At}\| \leq He^{-ht}$. This assumption and Lemma 2.1 lead to the definition of the following function

$$u(t) = e^{(h-\alpha)t}\|y(t)\| \leq k\varphi(0) + K\sum_{j=1}^{n}\|B_j\|\int_0^{\tau_i} e^{(h-\alpha)a}\|\varphi(a-\tau_j)\|da + KP\int_0^t u(a)da + KP\int_0^t e^{(h-\alpha)a}\sum_{i=1}^{n}\|v_i(a)\|da. \quad (36)$$

Since, $v_i(a) = y_i(a - \tau_i) = e^{(\alpha-h)(a-\tau_i)}$ so it is obvious that if $0 \leq M, \|y(a)\|, \|y(a - \tau_1)\|, \ldots, \|y(a - \tau_n)\| < \delta$ for $a \in [0, t]$ and $t \geq 0$ then

$$u(t) \leq M + +KP\int_0^t u(a)da + KP\int_0^t e^{(h-\alpha)\tau_i} u(a-\tau_i)da, \quad (37)$$

where M is defined in Equation (33). Let $b = max\{M, \varphi\}$, $b_0 = KP$ and $b_i = KPe^{(h-\alpha)\tau_i}$, $i = 1, 2 \ldots, n$ then define a function

$$\omega = b + b_0\int_0^t u(a)da + \sum_{i=1}^{n} b_i\int_0^t u(a-\tau_i)da \geq u(t). \quad (38)$$

Suppose for a convineint choose of $\varphi(t)$ and $\tau = max[\tau_i]$, then the function ω is defined in such way that $u(t - \tau_i) \leq 2\omega$. Thus, the given inequality holds

$$\omega'(t) = b_0\omega(t) + 2\sum_{i=1}^{n} b_i\omega(t) \leq 2\sum_{i=1}^{n} b_i\omega(t), \omega(0) = b. \quad (39)$$

Hence, $\|y(t)\| \leq be^{\left(\alpha+2\sum_{i=1}^{n}c_i-h\right)t}$. So if $\varphi(t), \tau$ and are choose such that $b < \delta$ and for a small P, the Equation (27) has stable solution.

Theorem 3.3 Suppose the assumptions of Theorem 3.1 are also satisfied. Let

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$f(y,v_1,v_2,...,v_n)=O\left(\|y\|^{\gamma^0}+\|v_1\|^{\gamma^1}+\|v_2\|^{\gamma^2}+...+\|v_n\|^{\gamma^n}\right)$ that is for all $P > 0$ there exists $\delta > 0$ such that if $\|y\|,\|v_1\|,\|v_2\|,...,\|v_n\|<\delta$ implies $\|f(y,v_1,v_2,...,v_n)\|<P\left(\|y\|^{\gamma^0}+\|v_1\|^{\gamma^1}+\|v_2\|^{\gamma^2}+...+\|v_n\|^{\gamma^n}\right)$.

Then, the solution of Equation (27) is exponentially stable.

Proof

Also, Equation (27) satisfies Equation (2) then according to Corollary 2.1 its solution has the form

$$y(t)=X_A(t)\varphi(0)+\sum_{j=1}^{n}B_j\int_0^{\tau_i}X_A(t-a)\varphi(a-\tau_i)\,da$$
$$+\int_0^t X_A(t-a)f(y,v_1,v_2,...,v_n)\,da.$$

Now, from the given property of eigenvalues of A implies that there are some positive constants $h,H > 0$ such that $\|e^{At}\| \leq He^{-ht}$. This assumption together with Lemma 3.1 yield

$$\begin{aligned}u(t)=e^{(h-\alpha)t}\|y(t)\|\leq k\varphi(0)+K\sum_{j=1}^{n}\|B_j\|\int_0^{\tau_i}e^{(h-\alpha)a}\\ \times\|\varphi(a-\tau_j)\|da+KP\int_0^t e^{(h-\alpha)a}\left[\|y\|^{\gamma^0}+\sum_{i=1}^{n}\|v_i(a)\|^{\gamma^i}\right]da.\end{aligned} \tag{40}$$

Since, $u(t)=e^{(h-\alpha)t}\|y(t)\|$ therefore the following functions can be obtained as $\|v(s)\|^{\gamma^0}=e^{(\alpha-h)a\gamma^0}u(a),\|v_i(a)\|^{\gamma^i}=\|y_i(a-\tau_i)\|^{\gamma^i}$ $=u(a-\tau_i)e^{(\alpha-h)(a\gamma^i-\tau_i\gamma^i)}$. Now, it is obvious that if $0 \leq M, \|y(a)\|, \|y(a-\tau_1)\|, \|y(a-\tau_2)\|, ..., \|y(a-\tau_n)\| < \delta$ for some $a \in [0,t]$ and $t \geq 0$ then

$$u(t)\leq b\int_0^t\eta_0(a)u(a)^{\gamma^0}da+\sum_{i=1}^{n}\int_0^t\eta_i(a)u(a-\tau_i)^{\gamma^i}da, \tag{41}$$

where

$$\eta_0(t)=KPe^{(h-\alpha)(1-\gamma^0)t},\eta_i=KPe^{(h-\alpha)\left[t+\gamma^i(\tau_i-t)\right]}$$

For $i = 1...,n, b = max\{M, \|\varphi\|\}$ and M is defined in Equation (33).

Let the rihgt-hand side of Equation (41) be ω, therefore by nondegreasing nature of the sequence $\{\gamma 0, \gamma 1 ... \gamma n\}$ and the validity of the inequality Equation (39), then the following inequality also hold.

$$\begin{aligned}\omega'(t)&\leq\eta_0\omega^{\gamma^0}+\sum_{i=1}^{n}2^{\gamma^i}\eta_i(t)\omega(t)\\ &\leq\sum_{i=0}^{n}{}^{\gamma^i}\eta_i(t)\omega(t),\omega(0)=b\end{aligned} \tag{42}$$

Let $\mu_i = 2^{\gamma^i}\eta_i(t)$ and $\theta_i = Z^{\gamma^i}$, for $i = 1,2,...,n$. Then Equation (42) can be deduced to the following form

$$\begin{aligned}\omega'(t)&\leq\mu_0\theta_0(\omega(t))+\mu_1\theta_1(\omega(t))+\\ &...+\mu_n\theta_n(\omega(t)),\omega(0)=0.\end{aligned} \tag{43}$$

Integrating both side of Equation (43) lead to

$$\omega(t)\leq c+\sum_{i=1}^{n}\int_0^t\mu_i(a)\theta_i(\omega(a))\,da=G(t). \tag{44}$$

Hence, the result follows from Manuel Pinto'S inequality (1990) in such a case the following can be obtained

$$\begin{aligned}&W_n(G(t))\leq\sum_{i=1}^{n}\int_0^t\mu_i(a)da,W_n(u)=\int_{u_n}^{u}\frac{dz}{W_n(z)}\text{and}\\ &W_n(c_{n-1})=W_n(c_{n-2})+\|\lambda_{n-1}\|_{b_1}\leq W_n(c_{n-1})+\|\lambda_n\|_{b_1}\\ &=W_n(c_n)=\int_{(c_n)_n}^{c_n}\frac{dz}{W_n(z)}\leq\int_{(c_n)_n}^{\infty}\frac{dz}{W_n(z)}\\ &\Rightarrow W_n(c_{n-1})=\int_0^t\lambda_n(a)da\leq\int_{(c_n)_n}^{\infty}\frac{dz}{W_n(z)}+\int_{c_{n-1}}^{\infty}\frac{dz}{W_n(z)}\\ &\leq\sum_{i=1}^{2}\int_{(c_n)_i}^{\infty}\frac{dz}{W_i(z)}\leq\sum_{i=1}^{\infty}\int_{(c_n)_i}^{\infty}\frac{dz}{W_i(z)}\leq\infty.\end{aligned} \tag{45}$$

Oviously, $W(G(0)) = W(b) = 0$. So for sufficiently small P then

$$\|\mu\|:=\sum_{i=1}^{\infty}\mu_i(a)da<\sum_{i=1}^{n}\int_{(c_n)_i}^{\infty}\frac{dz}{W_i(z)}<\infty.$$

Thus, $W(G(t)) < \|\mu\|$ for all $t \geq 0$ and hence

$u(t) \le W(t) \le G(t) < W^{-1}(\|\mu\|) \coloneqq C < \infty$.

Now, If $C < \delta$ then $\|\varphi\| \le c < C$ therefore, $\|y(t)\| \le Ce^{-(h-\alpha)t}, t \ge 0$.

Example: Consider the population dynamics model given in the following equation

$$
\begin{aligned}
y_1'(t) &= \mu y_1(t) + \alpha_1 y_1(t-\tau) - \gamma_1 y_1(t) y_1(t) \\
&- s y_1(t) y_1(t-\tau) - s y_1(t) y_2(t-\tau) \\
y_2'(t) &= -\beta y_2(t) + \alpha_2 y_2(t-\tau) + \gamma_2 y_1(t) y_2(t) \\
&- s y_2(t) y_1(t-\tau) - s y_2(t) y_2(t-\tau),
\end{aligned} \tag{46}
$$

Where $\alpha_1, \alpha_2, \gamma_1, \gamma_2, s > 0$ and $0 < \mu < \beta$.

So it obvious that $A = \begin{pmatrix} -\mu & 0 \\ 0 & -\beta \end{pmatrix}$ such that the eigenvalues $-\beta < -\mu < 0$, $B = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ and the function $f: \mathbb{R}^4 \to \mathbb{R}^2$ define by

$$
\begin{aligned}
f(y, v) = (&-\gamma_1 y_1(t) y_2(t) - s y_1(t) v_1 - s y_1(t) v_2, \\
&\gamma_2 y_1(t) y_2(t) - s y_2(t) v_1 - s y_2(t) v_2).
\end{aligned}
$$

It is also easy to see that A and B are 2×2 pairwise permutable matrices and $\|f(y, v)\| = P(\|y\|^2 + \|v\|^2)$. Since $\|e^{At}\| \le Ke^{-\mu t}$ for all $t \ge 0$ so if $\alpha < \mu$ where α satisfies

$$
\begin{vmatrix} \alpha e^{\mu\tau} & 0 \\ 0 & \alpha e^{\beta\tau} \end{vmatrix} \le \alpha e^{\alpha\tau}. \tag{47}
$$

Therefore, according to Theorem 3.3 the solution of Equation (46) is exponentially stable.

CONCLUSION

In this work, using convolution Theorem, the inverse Natural transform for an n th-multiple of Heaviside unit step functions are derived. Based on this result, a closed-form formula for the representation of the solution of nonlinear systems of DDEs with single and multiple numbers of constant delays is successfully established. The closed-form representation formulation is used to develop a criterion for the exponential stability of the solution of diffferent form of nonlinear systems of DDEs.

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