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IMPROVE NUMERICAL METHOD FOR SOLVING OPTIMAL CONTROL PROBLEMS USING PREDICTOR-CORRECTOR METHOD

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ABSTRACT

*I*n this paper, the numerical solution of optimal control problems using predictor-corrector methods with a forward-backward sweep algorithm is presented. The algorithm is developed to solve the state variable forward in time and the adjoint system alongside the control functions backward in time. The starting values of the methods were obtained using the classical Runge-Kutta method. The proposed algorithm is implemented using MAPLE 18 software to avoid a computational error. Some numerical examples are presented to illustrate the accuracy, reliability, consistency, and effectiveness of the present method. Finally, the results obtained were compared with the exact solution, and it performed favorably.

Keywords: Control Variable, Forward-Backward Sweep Algorithm, Predictor-Corrector Method, Runge-Kutta Method, State Variable.

INTRODUCTION

The predictor-corrector methods uses two formulae: predictor and corrector formulae to solve ordinary differential equations. The solution to $x_{n+1}(t)$ be first estimated using the predictor formula. Using the single-step technique or multiple prior points, the value $x_{n+1}(t)$ is determined using the known solution at the previous point $(t_n, x_n(t))$ [1, 2]. The corrector is used after an estimate of $x_{n+1}(t)$ has been discovered, and it uses the estimated value of $x_{n+1}(t)$ on the right side of the equation (implicit formula) to compute a new, more accurate value for $x_{n+1}(t)$ on its left side. The prediction technique first calculates the value of $x_{n+1}(t)$, which is then used in the right hand side of the corrector formula, resulting in a better approximation of $x_{n+1}(t)$. The acquired value of $x_{n+1}(t)$ is once more replaced in the corrector formula to determine an even more accurate approximation of $x_{n+1}(t)$. This process is repeated until two iterations of $x_{n+1}(t)$ provides values that are very close to one another. Since there is no need to solve a nonlinear equation, the corrector equation, which is an implicit equation, is applied explicitly in this scenario. [3].

Many researchers presented articles on Linear Multi-step Methods (LMM) and recently, [6] presented a work on Modified Runge-Kutta method with convergence analysis for nonlinear stochastic differential equations with Holder continuous diffusion coefficient and reported that the new method can achieve the optimal order of convergence compared to the classical Euler-Maruyama method at a finite time T without any restriction on the step size.

Applied Mathematics is concerned with methods of determining the best control method for a dynamical system known as optimal control theory. The work in [5] gives the foundation for optimality known as the

Pontryagin Minimum Principle (PMP), and is one of the most significant contributions to optimal control theory. This ground-breaking result offered a thorough analysis of optimal control theory, including different numerical approaches for resolving optimal control problems. Also, [8] developed a robust method for optimal control problems governed by system of Fredholm integral equations in mechanics and established that Lagrange polynomials is introduced to transform the optimal control problems into a nonlinear programming problem, and the resulting equations were implemented by the optimization algorithm using Lagrange collocation approach.

In a simple optimal control problem for ordinary differential equations, $x(t)$ and $u(t)$ were used to represent the state variable and the control variable, respectively. As a result of change in the control variable, the control function changes over time and the state variable satisfies an ordinary differential equation that depends on the control variable. [6, 7]. Finding a piecewise continuous control $u(t)$ and the related state variable $x(t)$ in our basic case constitutes the optimal control problem, according to [9], as

$$J = \int_{t_0}^{t_n} f\left(t, x(t), u(t)\right) dt \tag{1}$$

which satisfies a given dynamical control system of the form

$$x'(t) = g\left(t, x(t), u(t)\right), \tag{2}$$

and initial condition

$$x(0) = x_0 \tag{3}$$

In this article, our aim is to provide numerical solution of an optimal control problems of ordinary differential equations using predictor-corrector method. We developed a Predictor-Corrector Method, (PCM) based on Adams-Moultons Methods and used classical Runge-Kutta method to approximate the previous values. The forward-backward algorithm solved the state and adjoint functions of the optimal control problems. We tested the method on a set of optimal control problems using the step size $h = 0.1$ within the interval $0 \le x \le 1$. Moreover, the work in [10] obtained the solution of ordinary differential equations optimal control problems using classical Runge-Kutta method. In this purpose, our work will serves as an improvement to this approach and give more accurate results.

The paper organized as follows. In Section 2, important preliminary definitions and theorems from optimal control and Linear multi-step methods were given. Section 3, explains the transformation of optimal control problems to linear multi-step method. While, the numerical algorithm for the proposed method contained in section 4. Also, section 5, demonstrates the efficiency and reliability of the proposed method using some numerical examples and the computational results. Finally, Section 6, contains the conclusion remark.

Preliminaries

In this section, we introduce some basic definitions, theorems, and preliminaries facts which are used throughout this paper.

Definition 1: [11] Optimal Control is the process of determining control and state trajectories for a dynamic system, over a period of time, in order to optimize a given performance index.

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Definition 2: [12] State Function is the set of functions used to describe the mathematical state of the system.

Definition 3: [12] The control or control function is an operation that controls the recording, processing, or transmission of data.

Definition 4: [12] Performance Index is a measure of the quality of the trajectory.

Theorem 1: [16] If $(x(t),u(t),t)$ is the minimizer of (1), then there exists an adjoint state λ for which the triple $(x(t),u(t),t)$ satisfies the optimality conditions in (2) for all $t \in [t_0, t_n]$, where the Hamiltonian H is defined by

$$H\left(t,x(t),u(t),\lambda(t)\right)=f\left(t,x(t),u(t)\right)+\lambda(t)g\left(t,x(t),u(t)\right) \quad (4)$$

Definition 5: [11] Let $x:I\to$ be continuous on I and differentiable at all but finitely points of I . Further, suppose that $x'(t)$ is continuous wherever it is defined. Then, we say $x(t)$ is piecewise differentiable.

Definition 6: [2] An explicit methods are those methods that used an explicit formula for calculating the value of the dependent variable at the next value of the independent variable. In an explicit method, the right hand side of the equation only has all the known quantities. Therefore, the next unknown value of the dependent variable, $x_{n+1}(t)$ is calculated by evaluating an expression of the form

$$x_{n+1}(t)=f\left(t_n,t_{n+1},x_n(t)\right) \quad (5)$$

Definition 7: [2] In an implicit method the equation used for computing $x_{n+1}(t)$ from the known quantities $t_n, x_n(t)$ and $t_{n+1}(t)$ has the form

$$x_{n+1}(t)=f\left(t_n,t_{n+1},x_n(t),x_{n+1}(t)\right) \quad (6)$$

here, the unknown $x_{n+1}(t)$ appears on both side of the equation.

Theorem 2: Lipschitz theorem [2] If $f(t,x(t))$ is a real function defined and continuous in $x_n(t)$, $t \in (-\infty, +\infty)$, where t_0 and b are finite, then exists a constant $L>0$ called Lipschitz constant such that for any two values $x(t)=x_1(t)$ and $x(t)=x_2(t)$

$$\left|f\left(t,x_1(t)\right)-f(t,x_2(t))\right|<L\left|x_1(t)-x_2(t)\right| \quad (7)$$

where $t \in (t_0, b)$, then for any $x(t_0)=x_0$, then the initial value problem has a unique solution for $t \in (t_0, b)$.

Theorem 3: Taylor's theorem in Two Variables [1] Suppose that f(t, x) and all its partial derivatives of order less than or equal to $(n+1)$ are continuous on domain $\Omega=\{(t,x): a \le t \le b, c \le x \le d\}$ and (t,x) and $(t+\alpha, x+\beta)$ are all belong to Ω, then

$$f\left(t+\alpha,x+\beta\right)=P_n(t,x)+R_n(t,x), \quad (8)$$

Where

$$P_n(t,x) = f(t,x) + \alpha \frac{\partial}{\partial t} f(t,x) + \beta \frac{\partial}{\partial x} f(t,x)$$
$$+ \left[\frac{\alpha^2}{2!} \frac{\partial^2}{\partial t^2} f(t,x) + \alpha\beta \frac{\partial^2}{\partial t \partial x} f(t,x) + \frac{\beta^2}{2!} \frac{\partial^2}{\partial x^2} f(t,x) \right] + \ldots \quad (9)$$
$$+ \frac{1}{n!} \sum_{i=0}^{n+1} \binom{n}{i} \alpha^{n-i} \beta^i \frac{\partial^n}{\partial t^{n-i} \partial x^i} f(t,x),$$
$$R_n(t,x) = \frac{1}{(n+1)!} \sum_{i=0}^{n+1} \binom{n+1}{i} \alpha^{n+1-i} \beta^i \frac{\partial^{n+1}}{\partial t^{n+1-i} \partial x^i} f(t,x). \quad (10)$$

The function $P_n(t,x)$ is called the nth Taylor polynomial in two variables for the function $f(t,x)$ about (t_0, x_0), and $R_n(t,x)$ is the remainder term associated with $P_n(t,x)$.

Definition 8: [2] Runge-Kutta methods have the high-order local truncation error of the Taylor methods but eliminate the need to compute and evaluate the derivatives of $f(t,x)$. This is the family of single-step explicit, numerical techniques for solving first-order ordinary differential equations and used to finding the approximation $x_{i+1}(t)$ at the mesh point t_{i+1} of the form

$$x_{n+1}(t) = x_n + h \sum_{i=0}^{r} w_i x_i'(t_i + \alpha_i h) \quad (11)$$

where r is the slopes, w_i is the weighted average and α_i is the increment of the step-size h. For the derivation of this methods, one need to consider Taylors Theorem in two variables.

Definition 9: [14] An m−step multi-step method for solving the initial value problem has a difference equation for finding the approximation $x_{i+1}(t)$ at the mesh point t_{i+1} represented by the following equation, where n is an integer greater than 1:

$$x_{n+1}(t) = \sum_{i=1}^{r} \alpha_i x_{n-i+1}(t) + \sum_{i=0}^{r} \beta_i x_{n-i+1}'(t) \quad (12)$$

where α and β are given real constant. Thus, if $\beta_0 = 0$, the linear multi-step method is known as explicit method and if $\beta_0 \neq 0$, the linear multi-step method is known as implicit method.

Definition 10: [14] The local truncation error for multi-step methods is defined analogously to that of one-step methods. If $x(t_{n+1})$, is the exact solution, and $x_{n+1}(t)$ is the approximate solution, then the local truncation error $T_{n+1}(t)$ is defined by

$$T_{n+1}(t) = x(t_{n+1}) - x_{n+1}(t)$$
$$= x(t_{n+1}) - \sum_{i=1}^{r} \alpha_i x_{n-i+1}(t) - h \sum_{i=0}^{r} \beta_i x_{n-i+1}'(t) \quad (13)$$

Our goal is to solve such problems numerically, that is, develop an algorithm that generates an approximation to an optimal piecewise continuous control $u(t)$. For equal intervals of width h of the independent variable by performing iterations till the desired level of accuracy is obtained. In general, we divide the interval $[a, b]$ on which the solution is derived into finite number of sub intervals by the points

$a = t_0 < t_1 < t_2 < t_3, \ldots < t_n = b$, called the mesh points. This done by setting up $t_n = t_0 + nh$; these points will usually be equally spaced [11].

Transformation of Optimal Control Problems to Linear Multi-step Method

For this purpose, Adams-Bashforth and Adams-Moulton Predictor-Corrector Methods will be adopted for the solution of optimal control problem (1)−(3). Therefore, $x(t)$ and $\lambda(t)$ are the vector approximations for the state and adjoint respectively. We first make an initial guess for $u(t)$ over the given interval, and using the initial condition $x_1 = x(t_0)$ and the values for $u(t)$, solve $x(t)$ forward in time according to its differential equation in the optimality system, and then using the transversality condition $\lambda_{n+1}(t) = \lambda(t_1)$ and the values for $u(t)$, and $x(t)$, solve $\lambda(t)$ backward in time according to its differential equation in the optimality system. Finally, update $u(t)$ by entering the new $x(t)$ and $\lambda(t)$ values into the characterization of the optimal control [11].

Consider an approximation $x(t+h)$ of the dynamical control system of the form of an ordinary differential equation with step size h as follows:

$$x'(t) = f\left(t, x\left(t\right)\right), x(t_0) = x_0. \tag{14}$$

Now, integrating (15) from t_n to t_{n+1} , we get

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f\left(z, x(z)\right) dz \tag{15}$$

To evaluate the integrand in (15), $f\ (z, x(z))$ is approximated by a polynomial that interpolate $f\ (z, x(z))$ at r points $(t_n, x_n(t)), (t_{n-1}, x_{n-1}(t)), (t_{n-2}, x_{n-2}(t)), \ldots, (t_{n-r}, x_{n-r}(t))$, and applying Newton's Backward difference of degree $(r - 1)th$, with $t = t_n + h\tau$, we obtain

$$x(t_{n+1}) = x(t_n) + h\sum_{i=0}^{r}\int_0^1 \nabla^i f_n (-1)^i \binom{-\tau}{i} dr + T_r^{(0)} \tag{16}$$

where $T_r^{(0)}$ is the reminder term

$$T_r^{(0)} = h^{r+1}\int_0^1 (-1)^i \binom{-\tau}{i} f^r(\eta) dr. \tag{17}$$

Again, integrating (14) from t_{n-k} to t_{n+1} , we get

$$x(t_{n+1}) = x(t_{n-k}) + \int_{t_{n-k}}^{t_{n+1}} f\left(z, x(z)\right) dz \tag{18}$$

To evaluate the integrand in (18), $f\ (z, x(z))$ approximated by a polynomial that interpolate $f\ (z, x(z))$ at r points, that is $(t_{n+1}, x_{n+1}(t)), (t_n, x_n(t)), (t_{n-1}, x_{n-1}(t)), \ldots, (t_{n-r+1}, x_{n-r+1}(t))$, and applying Newton's Backward difference of degree $(r - 1)th$, with $t = t_n + h\tau$, we obtain

$$x(t_{n+1}) = x(t_{n-k}) + h\sum_{i=0}^{r}\int_{-k}^1 \nabla^i f_{n+1} (-1)^i \binom{1-\tau}{i+1} dr + T_r^{(k)} \tag{19}$$

where $T_r^{(k)}$ is the reminder term

$$T_r^{(k)} = h^{r+2}\int_{-k}^1 (-1)^{i+1} \binom{1-\tau}{i+1} f^{r+1}(\eta) dr. \tag{20}$$

Numerical Algorithm

In this section, a step-by-step algorithm was carried out to obtain a standard and more reliable scheme of the predictor-corrector methods for optimal control problems in ordinary differential equations. We first obtained the modified classical Runge-Kutta method, which serves as a single-step algorithm, and then proceeded to obtain the general scheme for the method developed. This work was developed using Maple 18 software for error-free and easy computations.

Step 1: Input, endpoints a, b; integer N; initial condition $x_i = x(t_0) = x_0$.

Step 2: Set the step size $h = (b - a)/N$; and $t_0 = a$; then Output (t_0, x_0).

Step 3: Set For i from 0 to 3 do step4 and use the initial guess of u_0 to solve $x(t)$ forward in time according to its differential equation in the optimality system transformed using modified Classical Runge-Kutta method.

Step 4: Set

$$\begin{aligned}
k_1 &= hf\left(t_i, x(t_i), u(t_i)\right);\\
k_2 &= hf\left(t_i + \frac{h}{2}, \frac{1}{2}\left(u_i(t) + u_{i+1}(t)\right), x(t_i) + \frac{k_1}{2})\right);\\
k_3 &= hf\left(t_i + \frac{h}{2}, \frac{1}{2}\left(u_i(t) + u_{i+1}(t)\right), x(t_i) + \frac{k_2}{2})\right);\\
k_4 &= hf\left(t_i + h, \frac{1}{2}\left(u_i(t) + u_{i+1}(t)\right), x(t_i) + k_3)\right);\\
x_{i+1}(t) &= x_i(t) + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right).
\end{aligned} \tag{21}$$

Step 5: end do:
Step 6: Set $j = 3 - i$
Step 7: Set For i from 0 to 3 do **step 8** and solve the transversality condition $\lambda_{N+1} = \lambda(t_1)$ using the values of $u(t)\ and\ x(t)$ backward in time according to its differential equation in the optimality system transformed using modified Classical Runge-Kutta method.
Step 8: Set

$$\begin{aligned}
l_1 &= hg\left(t_j, \lambda(t_i), x(t_j), u(t_j)\right);\\
l_2 &= hf\left(t_j - \frac{h}{2}, \frac{1}{2}\left(u_j(t) - u_{j-1}(t)\right), \frac{1}{2}\left(x_j(t) - x_{j-1}(t)\right), \lambda(t_j) - \frac{l_1}{2})\right);\\
l_3 &= hg\left(t_j - \frac{h}{2}, \frac{1}{2}\left(u_j(t) - u_{j-1}(t)\right), \frac{1}{2}\left(x_j(t) - x_{j-1}(t)\right), \lambda(t_j) - \frac{l_2}{2})\right);\\
l_4 &= hg\left(t_j - h, \frac{1}{2}\left(u_j(t) - u_{j-1}(t)\right), \frac{1}{2}\left(x_j(t) - x_{j-1}(t)\right), \lambda(t_j) - l_3)\right);\\
\lambda_{i+1}(t) &= \lambda_i(t) + \frac{1}{6}\left(l_1 + 2l_2 + 2l_3 + l_4\right).
\end{aligned} \tag{22}$$

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Step 9: end do:
Step 10: Set $t_i = t_0 + ih$.
Step 11: For i from 4 $to\ N$ do **step 12** to **step 13**
Step 12: Substitute for $r = 3$, into (17) to obtain Adams-Bashforth (explicit) method of the form

$$f_i(t) = f\left(t_i, x_i(t), \frac{1}{2}\left(u_i(t) + u_{i+1}(t)\right)\right);$$

$$f_{i-1}(t) = f\left(t_{i-1}, x_{i-1}(t), \lambda_{i-1}(t), \frac{1}{2}\left(u_{i-1}(t) + u_i(t)\right)\right);$$

$$f_{i-2}(t) = f\left(t_{i-2}, x_{i-2}(t), \lambda_{i-2}(t), \frac{1}{2}\left(u_{i-2}(t) + u_{i-1}(t)\right)\right);$$

$$f_{i-3}(t) = f\left(t_{i-3}, x_{i-3}(t), \lambda_{i-3}(t), \frac{1}{2}\left(u_{i-3}(t) + u_{i-2}(t)\right)\right);$$

$$x_{i+1}^{P}(t) = x_i(t) + \frac{h}{24}\left(55 f_i(t) - 59 f_{i-1}(t) + 37 f_{i-2}(t) - 9 f_{i-3}(t)\right).$$

Step 13: Substitute for $k = 0\ and\ r = 3$, into (20) to obtain Adams-Moulton Method (implicit) method of the form

$$f_{i+1}(t) = f\left(t_{i+1}, \frac{1}{2}\left(x_{i+1}(t) + x_{i+2}(t)\right), \frac{1}{2}\left(\lambda_{i+1}(t) + \lambda_{i+2}(t)\right), \frac{1}{2}\left(u_{i+1}(t) + u_{i+2}(t)\right)\right);$$

$$f_i(t) = f\left(t_i, \frac{1}{2}\left(x_{i+1}(t) + x_i(t)\right), \frac{1}{2}\left(\lambda_{i+1}(t) + \lambda_i(t)\right), \frac{1}{2}\left(u_{i+1}(t) + u_i(t)\right)\right);$$

$$f_{i-1}(t) = f\left(t_{i-1}, \frac{1}{2}\left(x_i(t) + x_{i+1}(t)\right), \frac{1}{2}\left(\lambda_i(t) + \lambda_{i+1}(t)\right), \frac{1}{2}\left(u_i(t) + u_{i+1}(t)\right)\right);$$

$$f_{i-2}(t) = f\left(t_{i-2}, \frac{1}{2}\left(x_{i-1}(t) + x_{i-2}(t)\right), \frac{1}{2}\left(\lambda_{i-1}(t) + \lambda_{i-2}(t)\right), \frac{1}{2}\left(u_{i-1}(t) + u_{i-2}(t)\right)\right);$$

$$x_{i+1}^{C}(t) = x_i(t) + \frac{h}{24}\left(9 f_{i+1}(t) + 19 f_i(t) - 5 f_{i-1}(t) + f_{i-2}(t)\right).$$

Step 14: Output $\left(x_i^P(t), x_i^C(t)\right)$ and set for next iterations.
Step 15: Set $j = N + 2 - i$.
Step 16: Repeat **step 11** to **step 14** for the Backward swept algorithm.

Numerical Examples

In this section, three different numerical examples were given to show the efficiency of our proposed method for approximating the solution of optimal control problems. In all the examples, the step size, h, is set to be 1.

Example 1. Find the optimal solutions that minimize objective function

$$\min \int_0^1 u^2(t)dt,$$

subject to dynamical condition $x'(t) = x(t) + u(t), \quad x(0) = 1,$

and transversality condition $\lambda(1)=0$.

with the exact solution $\left(x(t),u(t)\right)=\left(e^t,\dfrac{\lambda(t)}{2}\right)$;

Table 1: Absolute error of the state function for example 1

I	t_i	$x(t_i)$	x_i^p	$\lvert x(t_i)-x_i^p\rvert$	x_i^c	$\lvert x(t_i)-x_i^c\rvert$
5	0.5	1.64872	1.64873	1×10^{-5}	1.64873	1×10^{-5}
6	0.6	1.82212	1.82213	1×10^{-5}	1.82213	1×10^{-5}
7	0.7	2.01375	2.01376	1×10^{-5}	2.01376	1×10^{-5}
8	0.8	2.22554	2.22555	1×10^{-5}	2.22555	1×10^{-5}
9	0.9	2.45960	2.45961	1×10^{-5}	2.45961	1×10^{-5}
10	1	2.71820	2.71821	1×10^{-5}	2.71821	1×10^{-5}

Example 2. Consider the minimization problem of the form

$$\min\int_0^1\left(x^2(t)+\frac{1}{2}u^2(t)\right)dt,$$

subject to dynamical condition $x'(t)=x(t)+u(t),\quad x(0)=1,$

and transversality condition $\lambda(1)=0$.

with the exact solution $\left(x(t),u(t)\right)=\left(\dfrac{e^{1-t}}{2}-\dfrac{(e-4)}{2}e^t-1,1-e^{1-t}\right)$;

Table 2: Absolute error of the state function for example 2

I	t_i	$x(t_i)$	x_i^p	$\lvert x(t_i)-x_i^p\rvert$	x_i^c	$\lvert x(t_i)-x_i^c\rvert$
5	0.5	0.88096	0.88096	0	0.88096	0
6	0.6	0.91363	0.91364	1×10^{-5}	0.91364	1×10^{-5}
7	0.7	0.96546	0.96547	1×10^{-5}	0.96547	1×10^{-5}
8	0.8	1.03696	1.03697	1×10^{-5}	1.03697	1×10^{-5}
9	0.9	1.12884	1.12886	1×10^{-5}	1.12885	1×10^{-5}
10	1	1.24204	1.24205	1×10^{-5}	1.24204	0

Table 3: Absolute error of the Adjoint function for the example 2

I	t_i	$\lambda(t_i)$	λp_i	$\lvert\lambda(t_i)-\lambda p_i\rvert$	λ_i^c	$\lvert\lambda(t_i)-\lambda_i^c\rvert$
5	0.5	0.64872	0.64873	1×10^{-5}	0.64873	1×10^{-5}
4	0.4	0.82212	0.82213	1×10^{-5}	0.82213	1×10^{-5}
3	0.3	1.01375	1.01376	1×10^{-5}	1.01376	1×10^{-5}
2	0.2	1.22554	1.22555	1×10^{-5}	1.22555	1×10^{-5}
1	0.1	1.45960	1.45961	1×10^{-5}	1.45961	1×10^{-5}
0	0	1.71828	1.71829	1×10^{-5}	1.71829	1×10^{-5}

Example 3. Find the optimal solutions that minimize objective function

$$\min \frac{1}{2}\int_0^1 \left(3x^2(t)+u^2(t)\right)dt,$$

subject to dynamical condition $x'(t)=x(t)+u(t), \quad x(0)=1,$

and transversality condition $\lambda(1)=0.$

with the exact solution $\left(x(t),u(t)\right)=\left(\frac{3e^{-4}}{3e^{-4}+1}e^{2t}+\frac{3}{3e^{-4}+1}e^{2t}, \frac{3e^{-4}}{3e^{-4}+1}e^{2t}-\frac{3}{3e^{-4}+1}e^{2t}\right);$

Table 4: Absolute error of the state function for the example 3

i	t_i	$x(t_i)$	xp_i	$\lvert x(t_i)-x^p{}_i\rvert$	xc_i	$\lvert x(t_i)-x^c{}_i\rvert$
5	0.5	0.49030	0.49030	0	0.49032	2×10^{-5}
6	0.6	0.45843	0.45842	1×10^{-5}	0.45845	2×10^{-5}
7	0.7	0.44497	0.44494	3×10^{-5}	0.44498	1×10^{-5}
8	0.8	0.44936	0.44932	4×10^{-5}	0.44937	5×10^{-5}
9	0.9	0.47179	0.47173	6×10^{-5}	0.47179	0
10	1	0.51315	0.51307	8×10^{-5}	0.51314	1×10^{-5}

Table 5: Absolute error of the Adjoint function for the example 3

I	t_i	$\lambda(t_i)$	λp_i	$\lvert\lambda(t_i)-\lambda^p{}_i\rvert$	λ_i^c	$\lvert\lambda(t_i)-\lambda_i^c\rvert$
5	0.5	0.90457	0.90456	1×10^{-5}	0.90457	0
4	0.4	1.16186	1.16183	3×10^{-5}	1.16185	1×10^{-5}
3	0.3	1.46578	1.46573	5×10^{-5}	1.46576	2×10^{-5}
2	0.2	1.82852	1.82845	7×10^{-5}	1.82850	2×10^{-5}
1	0.1	2.26464	2.26454	1×10^{-4}	2.26426	3.8×10^{-4}
0	0	2.79166	2.79151	1.5×10^{-4}	2.79162	4×10^{-5}

CONCLUSION

In this paper, an improved method of solving optimal control systems is presented using predictor-corrector methods. Using the concept of the Adams-Moulton method, a predictor-corrector for the solution of an optimal control system is developed and approximated with the help of the classical Runge-Kutta method. Upon formulation of the method, an algorithm is developed using Maple 18 to obtain the numerical solution of the optimal control problem.

Suitable examples of optimal control problems were given, and the developed numerical algorithm was used to solve the optimal control problem. The solutions are compared with the exact solution, and it shows that the obtained solutions performed favorably. Table 1, for example, shows that there were no errors at the final time, and similarly, from Table 2 to Table 5, the numerical scheme and the exact solutions were in good agreement.

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