Bima Journal of Science and Technology, Vol. 7 (1) Mar, 2023 ISSN: 2536-6041

DOI: 0.56892/bima.v7i01.393

ENHANCED EFFICIENT ANALYTICAL APPROACH FOR NON-LINEAR SYSTEM OF DDES

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ABSTRACT

Delay differential equations (DDEs) have been used as fundamental tools in describing models in Sciences, engineering, and many other fields. These models in turn play a vital role in human life setting. However, analytical treatment of such models described by various forms of nonlinear systems of DDEs are very difficult to handle due to the lack of direct and simplified approach of evaluating the nonlinear terms. In this work, an efficient analytical approach for nonlinear systems of DDEs has been enhanced by modifying the He's polynomial. The aim is to ease the computational difficulties of nonlinear terms for the system of DDEs. The approach is applied to obtain approximate analytical solution of famous models from mathematical physics and biology. Therefore, the method provides solution of these models in form of polynomial series. Moreover, using an optimum value of auxiliary parameters the more precise approximation is obtained from only three iterations number of terms. Figures are used to illustrate the correctness of the result based on the residual error functions. Hence, the technique provides an easiest and straightforward means of solving these models analytically. Thus, it can also be used in obtaining solutions of other types of DDEs.

Keywords: Natural transform, Homotopy analysis method, Modified He's polynomial, Nonlinear DDEs Models.

INTRODUCTION

Delay differential equations arise in numerous areas of applied sciences and play a vital role in mathematical modelling of real-life phenomena. Several problems from various fields of studies contain a delay element. The few of such include the biological species living together (Fatemeh and Mehdi, 2008), the dynamics model of prey-predator, which gives rise to delay Volterra integro-differential equations (Mohammad and Mehdi, 2008), and the problems in population dynamics that lead to the formation of logistic delay equation (Mehdi and Rezvan, 2010). Partly due to the nature of the infinite dimensional state they possess, methods of solving ODEs are not generally applied to DDEs. Therefore, the analytical solutions of these models are hardly ever available. Hence, they are mostly solved by numerical methods. However, in sciences and engineering models, one can easily determine the important variables and how they are related to others using an analytical solution. It is also used to measure the influence of input on the output based on reliable mathematical support. So, it is obvious that analytical solutions are more general and reliable in related problems.

In this paper, an efficient analytical approach for nonlinear systems of DDEs which was first introduced by Aminu and Maan (2019) has been enhanced by modifying the He's polynomial. The mothod is from the combination of Homotopy analysis method

(HAM) and Natural transform (NT). For more information on NT and HAM the reader can see Liao (1992), Fethi *et al.* (2012) and Belgacem and Silambarasan (2012), respectively. An approximate analytical solution of two important models is obtained. The first model considered is from mathematical physics, and the second one is from mathematical biology. The He's polynomial is modified to suit and ease the computational dificulties of nonlinear terms of such models. The subsequent sections give the modification of the He's polynomial, the analysis of the approach and lastly outline the procedure of the method in finding the approximate analytical solutions of these models.

MATERIALS AND METHODS

This section provides the derivation of our new method. Fistly, the modification of the He's polynomial is considered in order to bring out the concept of the method clearly.

The Modification of He's Polynomial

In this section, the derivation of the modified He's polynomial that can be used for the calculation of nonlinear terms is provided.

Definition 1 The modified He's polynomial with respect to nonlinear function $F(\boldsymbol{v})$ for $\boldsymbol{v} = (v_1, v_2, ..., v_N)$, $v_i = v_{i,0} + v_{i,1} + v_{i,2} + ... + v_{i,m}$ with $i = 1,2, ..., N$ is defined as follow:

$$H_m(\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_N) = \frac{1}{m!}\frac{\partial^m}{\partial q^m} F\left(\sum_{p=0}^{m} q^p \boldsymbol{v}_p\,(t)\right)|_{q=0}, m \geq 0, \quad (1)$$

where $\boldsymbol{v}_i = (v_{i,1}, v_{i,1}, ..., v_{im})$ for $i = 1,2, ..., N$ and $\boldsymbol{v}_p = (v_{1,p}, v_{2,p}, ..., v_{N,p})$.

Theorem 1 Let $F(\phi_\lambda(t,q))$ be a smooth nonlinear function for $\phi_\lambda(t,q) = (\phi_1, \phi_2, ..., \phi_N)$ defined by $\phi_\lambda(t,q) = \sum_{p=0}^{\infty} \phi_{\lambda,p}(t,q) q^p$. If the embedding parameter $q = 1$, and the function $\phi_\lambda(t,1) = \boldsymbol{v}(t)$, with $\boldsymbol{v}(t) = (v_1(t), v_2(t), ..., v_N(t))$. Then the nonlinear function $F(\boldsymbol{v}_m)$ can be expressed as:

$$F(\boldsymbol{v}_m) = H_m(\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_N) = \frac{1}{m!}\frac{\partial^m}{\partial q^m} F\left(\sum_{p=0}^{m} q^p \boldsymbol{v}_p\right)|_{q=0}, m \geq 0, \quad (2)$$

where $\boldsymbol{v}_m = (v_{1,m}, v_{2,m}, ..., v_{N,m})$.

Proof

Since $\phi_\lambda(t,q) = \left(\sum_{p=0}^{\infty} \phi_{\lambda,p}(t,q)\right) q^p$, then according Ghorbani (2009) the following equation hold.

$$\frac{\partial^m}{\partial q^m} F(\phi_\lambda(t,q))|_{q=0} = \frac{\partial^m}{\partial q^m} F\left(\left(\sum_{p=0}^{\infty} \phi_{\lambda,p}(t,q)\right) q^p\right)|_{q=0}$$

$$= \frac{\partial^m}{\partial q^m} F\left(\left(\sum_{p=0}^{m} \phi_{\lambda,p}(t,q)\right) q^p\right)|_{q=0}. \quad (3)$$

The Maclaurin series expansion of $F(\phi_\lambda(t,q))$ with respect to q is given as:

$$F(\phi_\lambda(t,q)) = F(\phi_\lambda(t,q))|_{q=0} + \left(\frac{\partial}{\partial q} F(\phi_\lambda(t,q))|_{q=0}\right) q + \frac{1}{2!}\left(\frac{\partial^2}{\partial q^2} F(\phi_\lambda(t,q))|_{q=0}\right) q^2 \frac{1}{3!}\left(\frac{\partial^2}{\partial q^2} F(\phi_\lambda(t,q))|_{q=0}\right) q^3 + \ldots + \frac{1}{m!}\left(\frac{\partial^2}{\partial q^2} F(\phi_\lambda(t,q))|_{q=0}\right) q^m + \ldots \quad (4)$$

The hypothesis of the theorem implies the following result.

$$F(\phi_\lambda(t,q)) = F\left(\sum_{p=0}^{\infty} \phi_{\lambda,p}(t,q)\, q^p\right)|_{q=0} + \left(\frac{\partial}{\partial q} F\left(\sum_{p=0}^{\infty} \phi_{\lambda,p}(t,q)\, q^p\right)|_{q=0}\right) q + \frac{1}{2!}\left(\frac{\partial^2}{\partial q^2} F\left(\sum_{p=0}^{\infty} \phi_{\lambda,p}(t,q)\, q^p\right)|_{q=0}\right) q^2 + \ldots$$

$$+\frac{1}{m!}\left(\frac{\partial^m}{\partial q^m}F\left(\sum_{p=0}^{\infty}\phi_{\lambda,p}(t,q)\,q^p\right)|_{q=0}\right)q^2 + \dots \quad (5)$$

According to Equations (3), Equation (5) leads to:

$$F(\phi_\lambda(t,q)) = F(\phi_{\lambda,0}(t,q))|_{q=0} + \left(\frac{\partial}{\partial q}F\left(\sum_{p=0}^{1}\phi_{\lambda,p}(t,q)\,q^p\right)|_{q=0}\right)q + \frac{1}{2!}\left(\frac{\partial^2}{\partial q^2}F\left(\sum_{p=0}^{2}\phi_{\lambda,p}(t,q)\,q^p\right)|_{q=0}\right)q^2 + \dots$$

$$+\frac{1}{m!}\left(\frac{\partial^m}{\partial q^m}F\left(\sum_{p=0}^{m}\phi_{\lambda,p}(t,q)\,q^p\right)|_{q=0}\right)q^m + \dots (6)$$

The assumption of theorem yields the following result.

$$F(\boldsymbol{v}) + F(\boldsymbol{v}_0)|_{q=0} + \frac{\partial}{\partial q}F\left(\sum_{p=0}^{1}\boldsymbol{v}_p\,q^p\right)|_{q=0} + \frac{1}{2!}\frac{\partial^2}{\partial q^2}F\left(\sum_{p=0}^{2}\boldsymbol{v}_p\,q^p\right)|_{q=0} + \dots + \frac{1}{m!}\frac{\partial^m}{\partial q^m}F\left(\sum_{p=0}^{m}\boldsymbol{v}_p\,q^p\right)|_{q=0} + \dots \quad (7)$$

Therefore, at m^{th} −order derivative the nonlinear function $F(v_m)$ can be obtained from the following relation

$$F(\boldsymbol{v}_m) = \frac{1}{m!}\frac{\partial^m}{\partial q^m}F\left(\sum_{p=0}^{m}q^p\boldsymbol{v}_p\right)|_{q=0} = H_m(\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_N) \quad (8)$$

Hence the proof.

Enhanced Efficient Analytical Approach for Nonlinear System of DDEs

In this section, the modification of the method given by Aminu and Maan (2019) is presented. Now, consider the following n-order nonlinear system of Retarded (RDDEs) of the form:

$$v^{(n)}(t) = f(t, v(t), v^{'}(t), \dots, v^{(n-1)}(t), v(\mu(t)), v^{'}(\mu(t)), \dots, v^{(n-1)}(\mu(t)), t \in [0,\infty), j = 1,2,\dots,r, \quad (9)$$

where $\boldsymbol{v}^{(n)}(t) = \left[v_1^{(n)}(t), v_2^{(n)}(t), \dots, v_N^{(n)}(t)\right]^T$, $\boldsymbol{f} = [f_1, f_1, \dots, f_N]^T$ such that $f_i: [0,\infty) \times \mathbb{R}^{nN} \times \mathbb{R}^{w} \to \mathbb{R}$ are continuous functions for $i = 1,2,\dots,N, w = nNr$. $\boldsymbol{v}^{(k)}(t) = \left[v_1^{(k)}(t), v_2^{(k)}(t), \dots, v_N^{(k)}(t)\right]^T$, $\boldsymbol{v}^{(k)}(\mu_j(t)) = \left[v_1^{(n)}(\mu_j), v_2^{(n)}(\mu_j), \dots v_N^{(n)}(\mu_j)\right]^T$ for $k = 1,2,\dots,n-1$ and $\mu_j(t)$ are continuous delay functions for $j = 1,2,\dots,r$. In this research the delay functions $\mu_j(t)$are defined as:

$(i)\mu_j = a_j$ where $a_j \in (0,1)$ (proportional delay)

$(ii)\ t - \tau_j$ where $\tau_j > 0$ is real constant (constant delay) for $j = 1,2,\dots,r$.

The initial condition is defined as

$$v_i(0) = u_{i,0}, v_i^{'}(0) = u_{i,1}, \dots, v_i^{(n-1)}(0) = u_{i,n-1},$$

$$v_i(t) = p_i(t)\ t \in [-D, 0], i = 1,2,\dots,N, \quad (10)$$

where $u_{i,0}, u_{i,1}, \dots, u_{i,n-1}$ are constants real numbers, $D = \max_{j=1,2,\dots,r}[\tau_j]$ and $p_i, p_i^{'}, \dots, p_i^{(n-1)}$ are continuous on $[-D, 0]$. Equation (9) can be written in the following form to separate the linear and nonlinear terms of the equation.

$$L_i(v_i) + R_i(\boldsymbol{v}) + F_i(\boldsymbol{v}) = g_i(t), \quad (11)$$

with Equation (10) as initial condition and $\boldsymbol{v} = (v_1, v_2, \dots, v_N)$ The linear part is decomposed into $L_i + R_i$ where $L_1, L_2, \dots, L_N$ are linear operators, $R_1, R_2, \dots, R_N$ are remainders of the linear operators, and $F_1, F_2, \dots, F_N$ are nonlinear functions represent the nonlinear terms. Take the Natural transform of both sides of Equation (11) to obtain the simplified form of Equation (9) as:

$$\boldsymbol{N}^{+}[L_i(v_i)+R_i(\boldsymbol{v})+F_i(\boldsymbol{v})]=\boldsymbol{N}^{+}[g_i(t)], \quad (12)$$

where $\boldsymbol{N}^{+}$ denotes Natural transform. By Substituiting Equation (10) into Equation (12) and using the differential properties of Natural transform the simplify form of Equation (9) is obtained as:

$$\boldsymbol{N}^{+}[v_i]-\frac{u^n}{s^n}\sum_{k=1}^{n}\frac{s^{n-k}}{u^{n-k+1}}v_i^{k-1}(0)+\frac{u^n}{s^n}\boldsymbol{N}^{+}[R_i(\boldsymbol{v})+F_i(\boldsymbol{v})-g_i(t)]=0. \quad (13)$$

Based on Equation (13) a nonlinear operator is defined as

$$\boldsymbol{N_i}(\phi_i(t,q))=\boldsymbol{N}^{+}[\phi_i(t,q)]-\frac{u^n}{s^n}\sum_{k=1}^{n}\frac{s^{n-k}}{u^{n-k+1}}\phi_i^{k-1}(0)+\frac{u^n}{s^n}\boldsymbol{N}^{+}[R_i[\phi_\lambda(t,q)]+F_i[\phi_\lambda(t,q)]-g_i(t)], \quad (14)$$

where $q\in[0,1]$ is an embedding parameter, $\phi_\lambda(t,q)=(\phi_1,\phi_2,\dots,\phi_N)$ is a function of real variables t and q. Then by the means of HAM the recursive relation of Equations (9)-(10) is obtain as:

$$v_{i,m}(t)=(\chi_m+\hbar_i)v_{i,m-1}(t)+\hbar_i(1-\chi_m)\times\boldsymbol{N}^{-}\left(\sum_{k=1}^{n}\frac{u^{k-1}}{s^k}v_i^{k-1}(0)+\frac{u^n}{s^n}\boldsymbol{N}^{+}[g_i(t)]\right)+\hbar_i\times \quad (15)$$

$$\boldsymbol{N}^{-}\left\{\frac{u^n}{s^n}\boldsymbol{N}^{+}\left[R_i(\boldsymbol{v}_{m-1}(t))+H_{i,m}(\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_N)\right]\right\}, m\geq 1$$

where $\boldsymbol{v}_{m-1}(t)=(v_{1,m-1},v_{2,m-1},\dots,v_{N,m-1})$ and nonlinear function $F_i(\boldsymbol{v})$ is computed using the modified He's polynomial $H_{i,m}(\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_N)$ defined as

$$H_m(\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_N)=\frac{1}{m!}\frac{\partial^m}{\partial q^m}F\left(\sum_{p=0}^{m}q^p\boldsymbol{v}_p(t)\right)|_{q=0}, m\geq 0, \quad (16)$$

where $\boldsymbol{v}_i=(v_{i,1}\dots,v_{im})$ for $i=1,2,\dots,N$ and $\boldsymbol{v}_p=(v_{1,p},v_{2,p},\dots,v_{N,p})$.

Convergence Analysis

In this section, some necessary and sufficient conditions are established to confirm the convergence of the schematic technique derived in above section. In such a case, it is required to determine the convergent of the series solution $\sum_{m=0}^{\infty}v_{i,m}(t)$ within a restricted subset of the domain of the problem, that is the region of convergence (Robert and Vajravelu., 2009). Thus, let $[0,T]$ be the region of convergence of the series solution and the convergence theorem is established through the use of Banach's fixed point theorem.

Theorem 2 Let $C([0,T],||.||)$ be a Banach space of continuous functions on $[0,T]$ and $X_i:[0,T]\to[0,T]$ be a mapping such that

$$\| X_i(v_i(t))-X_i(r_i(t))\|\leq\rho_i\| v_i(t)-r_i(t)\| \; v_i(t),r_i(t)\in C([0,T]), i=1,2,\dots,N, \quad (17)$$

for some $\rho_i\in(0,1)$. Suppose the functions $p_i,p_i^{'},\dots,p_i^{(n-1)}$ are continuous on $[-D,0]$ and f_i are continuous on $[0,T]$ and Lipschitz on $[0,T]\times\mathbb{R}^{nN}\times\mathbb{R}^{w}$. If the initial approximation $v_{i,0}(t)$ satisfy the initial condition $v_i(0),v_i^{'}(0),\dots,v_i^{(n-1)}(0)$ and $||v_{m+1}(t)||\leq\rho_i||v_m(t)||$ for all $m\geq m_0$ for some $m_0\in\mathbb{N}$. Then the series $\sum_{m=0}^{\infty}v_{i,m}(t)$ converges to the unique solution of Equations (9)-(10) in the interval $[0,T]$ with the norm defined as:

$$\| v_i(t)\|=\sup_{t\in[0,T]}|v_i(t)|, i=1,2,\dots,N. \quad (18)$$

DOI: 0.56892/bima.v7i01.393

Proof

Suppose $C([0,T],||.||)$ is a Banach space of continuous functions on $[0,T]$ and $S_{i,n} = \sum_{m=0}^{n} v_{i,m}(t)$ be the sequence of partial sum of the series $\sum_{m=0}^{\infty} v_{i,m}(t)$. Therefore, it is enough to show that $\{S_{i,n}\}_{n=0}^{\infty}$ is a Cauchy sequence in $C([0,T],||.||)$. So, consider the following equation.

$$\left|\left|S_{i,n+1} - S_{i,n}\right|\right| = \| \left(v_{i,0} + v_{i,1} + \ldots + v_{i,n+1}\right) - \left(v_{i,0} + v_{i,1} + \ldots + v_{i,n}\right)|| = \| v_{i,n+1} \| \leq \rho_i ||v_{i,n}|| \leq \rho_i^{'} ||v_{i,n-1}|| \leq \rho_i^{n+1}||v_{i,0}||. \quad (19)$$

Therefore, if $m, p \in \mathbb{N}$ with $m > p$, then, by the means of Equation (19) and triangular inequality the following equation holds.

$$\left|\left|S_{i,m} - S_{i,p}\right|\right| = \left|\left|\left(S_{i,m} - S_{i,m-1}\right) + \left(S_{i,m-1} - S_{i,m-2}\right) + \ldots + \left(S_{i,p+1} - S_{i,p}\right)\right|\right| \leq \left|\left|\left(S_{i,m} - S_{i,m-1}\right)\right\| + \left\|\left(S_{i,m-1} - S_{i,m-2}\right)\right\| + \ldots + \left\|\left(S_{i,p+1} - S_{i,p}\right)\right|\right| \leq \rho_i^m \left|\left|v_{i,0}(t)\right|\right| + \rho_i^{m-1}\left|\left|v_{i,0}(t)\right|\right| + \ldots + \rho_i^{p+1}\left|\left|v_{i,0}(t)\right|\right| \leq \sup_{t\in[0,T]} \left[\rho_i^m|v_{i,0}(t)| + \rho_i^{m-1}|v_{i,0}(t)| \ldots + \rho_i^{p+1}|v_{i,0}(t)|\right]$$

$$= \rho_i^{p+1} \sup_{t\in[0,T]} |v_{i,0}(t)| \sum_{i=0}^{m-p-1} \rho_i^{p+1} \leq \rho_i^{p+1} \sup_{t\in[0,T]} |v_{i,0}(t)| \times \sum_{i=0}^{\infty} \rho_i^{p+1}$$

$$= \rho_i^{p+1} \sup_{t\in[0,T]} |v_{i,0}(t)| \sum_{i=0}^{\infty} \left(\frac{1}{1-\rho_i}\right) \quad (20)$$

Since $\rho_i \in (0,1)$ so for arbitrary $\varepsilon_i > 0$ there exist $\eta_i \in \mathbb{N}$ such that $\rho_i^{\eta_i} < \frac{\varepsilon_i(1-\rho_i)}{\sup_{t\in[0,T]}|v_{i,0}(t)|}$.

Therefore, by choosing $m, p > \eta_i$ then Equation (20) yields the following result.

$$\left|\left|S_{i,n+1} - S_{i,n}\right|\right| \leq \rho_i^{p+1} \sup_{t\in[0,T]} |v_{i,0}(t)| \sum_{i=0}^{\infty} \left(\frac{1}{1-\rho_i}\right) < \frac{\varepsilon_i(1-\rho_i)}{\sup_{t\in[0,T]} |v_{i,0}(t)|} \sup_{t\in[0,T]} |v_{i,0}(t)| \sum_{i=0}^{\infty} \left(\frac{1}{1-\rho_i}\right) = \varepsilon_i \quad (21)$$

This shows that $\{S_{i,n}\}_{n=0}^{\infty}$ is a Cauchy sequence in $C([0,T],||.||)$. Hence, the serie $\sum_{m=0}^{\infty} v_{i,m}(t)$ converges to the solution of Equations (9)-(10) in the interval $[0,T]$. Since $C([0,T],||.||)$ is Banach space implies that $([0,T],||.||)$ is a complete norm space. Therefore, the series $S_{i,n}$ has a limit say $v_{i,}(t)$ in $[0,T]$. According to the hypothesis of the theorem, X_i is a contraction mapping and posseses a fixed point in $[0,T]$. Thus, if $\lim_{n\to\infty} S_{i,n} = v_i(t)$ such that $\sum_{m=0}^{\infty} v_{i,m}(t) = v_{i,}(t)$ then $v_{i,}(t)$ is a fixed point of X_i in $[0,T]$ as shown below using triangular inequality.

$$\left|\left|v_i(t) - X_i(v_i(t))\right|\right| = \left|\left|v_i(t) - S_{i,m} + S_{i,m} - X_i(v_i(t))\right|\right| \leq \left|\left|v_i(t) - S_{i,m}\right|\right| + \left|\left|S_{i,m} - X_i(v_i(t))\right|\right| = \left|\left|v_i(t) - S_{i,m}\right|\right| + \left|\left|X_i(S_{i,m}) - X_i(v_i(t))\right|\right| \leq ||v_i(t) - S_{i,m}|| + \rho_i \left|\left|S_{i,m-1} - v_i(t)\right|\right|. \quad (22)$$

From equation (22) as $m \to \infty$ then, $S_{i,m-1}, S_{i,m} \to v_i(t)$. This implies that $\left|\left|v_i(t) - X_i(v_i(t))\right|\right| \leq 0$. By the property of norm if $\left|\left|v_i(t) - X_i(v_i(t))\right|\right| \leq 0 \Rightarrow \left|\left|v_i(t) - X_i(v_i(t))\right|\right| = 0 \Rightarrow v_i(t) - X_i(v_i(t)) = 0$ $\Rightarrow v_i(t) = X_i(v_i(t))$. Hence, $v_i(t)$ is a fixed point of X_i that is $X_i(v_i(t)) = v_i(t)$.

RESULTS

In this section, the derived method is applied to obntain the approximate analytical solution of these models. The advanced Lorenz system is to be considered firrst, then the spread of infectious diseases model.

Advanced Lorenz System

The advanced Lorenz system is introduced by *Zhang et.al.* (2009), and it is an

DOI: 0.56892/bima.v7i01.393

advancement of the model proposed by Lorenz in (1963) for atmospheric convection. The projective synchronization for the pairs of time-delayed chaotic systems amongst advanced Lorenz system is investigated by Ansari and Das in (2015) employing of an adaptive control approach based on LyapunovKrasovskii functional theory.

According to Ansari and Das (2015) the system is given as follows:-

$$v_1'(t) = 20[v_2(t) - v_1(t)] + 3v_1(t-\tau)$$
$$v_2'(t) = 14v_1(t) + \frac{53}{5}v_2(t) - v_1(t)v_3(t) \quad (23)$$
$$v_3'(t) = v_1^2(t) - \frac{14}{5}v_3(t),$$

with initial condition

$$v_1(0) =- 20,\ v_2(0) = 8,\ v_3(0) = 20. \quad (24)$$

Following the above algorithm the Natural transform of both sides of Equation (23) should be first taken and simplify further using Equation (24) and diffierential properties of NT to obtain

$$\boldsymbol{N}^+[v_1(t)] + \frac{20}{s} + \boldsymbol{N}^+[v_1(t) - v_2(t)] - \frac{3u}{s}\boldsymbol{N}^+[v_1(t-\tau)] = 0$$
$$\boldsymbol{N}^+[v_2(t)] - \frac{8}{s} - \frac{14u}{s}\boldsymbol{N}^+[v_1(t)] - \frac{53u}{5s}\boldsymbol{N}^+[v_2(t)] + \frac{u}{s}\boldsymbol{N}^+[v_1(t)v_3(t)] = 0 \quad (25)$$
$$\boldsymbol{N}^+[v_3(t)] - \frac{20}{s} + \frac{14u}{5s}\boldsymbol{N}^+[v_3(t)] - \frac{u}{s}\boldsymbol{N}^+[v_1^2(t)] = 0$$

Now, define a nonlinear operator based on Equation (25) as:

$$\boldsymbol{N}(\phi_1(t,q)) = \boldsymbol{N}^+[\phi_1(t,q)] + \frac{20}{s} + \boldsymbol{N}^+[\phi_1(t,q) - \phi_2(t,q)] - \frac{3u}{s}\boldsymbol{N}^+[v_1(t-\tau)] = 0$$
$$\boldsymbol{N}(\phi_2(t,q)) = \boldsymbol{N}^+[\phi_2(t,q)] - \frac{8}{s} - \frac{14u}{s}\boldsymbol{N}^+[\phi_1(t,q)] - \frac{53u}{5s}\boldsymbol{N}^+[\phi_2(t,q)] + \frac{u}{s}\boldsymbol{N}^+[\phi_1(t,q)\phi_3(t,q)] = 0 \quad (26)$$
$$\boldsymbol{N}(\phi_3(t,q)) = \boldsymbol{N}^+[\phi_3(t,q)] - \frac{20}{s} + \frac{14u}{5s}\boldsymbol{N}^+[\phi_3(t,q)] - \frac{u}{s}\boldsymbol{N}^+[{\phi_1}^2(t,q)] = 0$$

Based on Equation (15) the recursive relation of the model is obtained as:

$$v_{1,m}(t) = (\chi_m + \hbar_1)v_{1,m-1} + \hbar_1(1-\chi_m)\boldsymbol{N}^-\left[\frac{20}{s}\right] + \hbar_1\boldsymbol{N}^-\left\{\frac{u}{s}\boldsymbol{N}^+[R_1(v_{1,m-1}, v_{2,m-1})]\right\}$$
$$v_{2,m}(t) = (\chi_m + \hbar_2)v_{2,m-1} - \hbar_2(1-\chi_m)\boldsymbol{N}^-\left[\frac{8}{s}\right] - \hbar_2\boldsymbol{N}^-\left\{\frac{u}{s}\boldsymbol{N}^+[R_2(v_{1,m-1}, v_{2,m-1})]\right\} + \hbar_2\boldsymbol{N}^-\left\{\frac{u}{s}\boldsymbol{N}^+[H_{2,m}(v_{1,1}, v_{3,1}, \ldots, v_{1,N}, v_{3,N})]\right\} \quad (27)$$

$$v_{3,m}(t) = (\chi_m + \hbar_3)v_{3,m-1} - \hbar_3(1-\chi_m)\boldsymbol{N}^-\left[\frac{20}{s}\right] + \hbar_3\boldsymbol{N}^-\left\{\frac{u}{s}\boldsymbol{N}^+[R_3(v_{3,m-1})]\right\} - \hbar_3\boldsymbol{N}^-\left\{\frac{u}{s}\boldsymbol{N}^+[H_{3,m}(v_{1,1}, v_{1,2}, \ldots, v_{1,N})]\right\}. \quad (27)$$

Here, the nonlinear functions $v_1(t), v_3(t)$ and $v_1^2(t)$ are respectively calculated using the modifiedHe's polynomials,

$$H_{2,m}(v_{1,1}, v_{3,1}, \ldots, v_{1,N}, v_{3,N}) = \frac{1}{m!}\frac{\partial^m}{\partial q^m}F_2\left(\sum_{p=0}^{m} q^p(v_{1,p}(t), v_{3,p}(t))\right)|_{q=0},$$
$$H_{3,m}(v_{1,1}, v_{1,2}, \ldots, v_{1,N}) = \frac{1}{m!}\frac{\partial^m}{\partial q^m}F_2\left(\sum_{p=0}^{m} v_{1,p}(t)q^p\right)|_{q=0}.$$

So, by choosing the initial approximation as $v_{1,0}(t) =- 20, v_{2,0}(t) = 8, v_{3,0}(t) =- 20$. Then from Equation (27), the terms of $v_{1,m}, v_{2,m},$ and $v_{3,m},$ are given as follows:-

$$v_{1,1}(t) = \hbar_1 v_{1,0}(t) + 20\hbar_1 + \hbar_1\boldsymbol{N}^-\left\{\frac{u}{s}\boldsymbol{N}^+[R_1(v_{1,0},(t)v_{2,0}(t))]\right\} = -20\hbar_1 + 20\hbar_1 - 5000\hbar_1 t =- 5000\hbar_1 t.$$
$$v_{1,2}(t) = (1+\hbar_1)v_{1,1}(t) + \hbar_1\boldsymbol{N}^-\left\{\frac{u}{s}\boldsymbol{N}^+[R_1(v_{1,1},(t)v_{2,1}(t))]\right\} = [2048\hbar_1\hbar_2 - 4250{\hbar_1}^2]t^2 - \frac{1}{2}[1003{\hbar_1}^2 + 1000\hbar_1]t$$
$$v_{1,3}(t) = (1+\hbar_1)v_{1,2}(t) +$$

$\hbar_1 \boldsymbol{N}^{-}\left\{\frac{u}{s}\boldsymbol{N}^{+}\left[R_1\left(v_{1,2},(t)v_{2,2}(t)\right)\right]\right\} = \frac{4}{5}\left[-96050{\hbar_1}^2 - 27136\hbar_1{\hbar_2}^2 - 86000\hbar_1\hbar_2\right]t^3 + \frac{4}{5}\left[4108{\hbar_1}^2\hbar_2 - 17051{\hbar_1}^3 + 8192\hbar_1\hbar_2 - 17000{\hbar_1}^2 + 4096\hbar_1{\hbar_2}^2\right]t^2 - \frac{1}{2}\left[1006{\hbar_1}^3 + 20003{\hbar_1}^2 + 1000\hbar_1\right]t.$

$v_{2,1}(t) = \hbar_2 v_{2,0}(t) - 8\hbar_2 - \hbar_2\boldsymbol{N}^{-}\left\{\frac{u}{s}\boldsymbol{N}^{+}\left[R_2\left(v_{1,0}(t),v_{2,0}(t)\right) - H_{2,0}\right]\right\} = 8\hbar_2 - 8\hbar_2 - \frac{1024}{5}\hbar_2 t = -\frac{1024}{5}\hbar_2 t$

$v_{2,2}(t) = (1+\hbar_2)v_{2,1}(t) - \hbar_2\boldsymbol{N}^{-}\left\{\frac{u}{s}\boldsymbol{N}^{+}\left[R_2\left(v_{1,1}(t),v_{2,1}(t)\right) - H_{2,1}\right]\right\} = -\frac{1024}{5}\left({\hbar_1}^2 + \hbar_2\right)t - \frac{1}{25}(375000\hbar_1\hbar_2 - 86000\hbar_2\hbar_3) \quad (28)$

$v_{2,3}(t) = (1+\hbar_2)v_{2,2}(t) - \hbar_2\boldsymbol{N}^{-}\left\{\frac{u}{s}\boldsymbol{N}^{+}\left[R_2\left(v_{1,2}(t),v_{2,2}(t)\right) - H_{2,2}\right]\right\} = \frac{1}{3}\left[23188\hbar_1{\hbar_2}^2 - 25500{\hbar_1}^2\hbar_2 - 1105.66{\hbar_2}^2\hbar_3 - 372000\hbar_1\hbar_3\right]t^3 + \frac{1}{25}\left[105472{\hbar_2}^3 + 105472{\hbar_2}^2 + 37612.5{\hbar_1}^2\hbar_2 - 375000\hbar_1{\hbar_2}^2 - 86000{\hbar_2}^2\hbar_3 - 86000\hbar_1\hbar_3\right]t^2 - \left(204.8{\hbar_2}^3 + 409.6{\hbar_2}^2 + 204.8\hbar_2\right)t \quad (29)$

Now from Equations (28) and (29) at $\hbar_1 = -0{:}997$ $\hbar_2 = -1$ and $\hbar_3 = -1$ the third-order approximation of this model is given as

$$v_1(t) \approx \sum_{m=0}^{3} v_{1,m}(t) = 32647.5t^3 - 2182.643t^2 + 498.4977t - 20$$

$$v_2(t) \approx \sum_{m=0}^{3} v_{2,m}(t) = -112116.7778t^3 + 3029.94t^2 + 204.8t + 8$$

$$v_3(t) \approx \sum_{m=0}^{3} v_{3,m}(t) = 122476.17t^3 + 1045.6t^2 + 2344t + 20 \quad (30)$$

Since the exact solution of the equation is not available, to know more about the behaviour of its solution requires considering the higher-order approximations. As such, by following the same process as in Equations (28) and (29) the fourth-order approximation is generated as follows:-

$$v_1(t) \approx \sum_{m=0}^{4} -69933.1934t^4 + 32647.9733t^3 - 2182.5373t^2 + 498.4978t - 20$$

$$v_2(t) \approx \sum_{m=0}^{4} 1765871.937t^4 - 112116.5906t^3 + 3029.9464t^2 + 204.8t + 8$$

$$v_2(t) \approx \sum_{m=0}^{4} -962709.313t^4 + 122476.17t^3 + 1045.5578t^2 + 2344t + 20 \quad (31)$$

Hence, the higher order approximation ($m \geq 5$) can be obtain in a similar way. Since the model has no exact solution, the accuracy of the approximate solutions can be measured by using the residual error functions defined as follows:-

$\boldsymbol{E}_i[v_1(t)] = \boldsymbol{V}'_{1,i}(t) - 20\left[\boldsymbol{V}_{2,i}(t) - \boldsymbol{V}_{1,i}(t)\right] - 3\boldsymbol{V}_{1,i}(t-\tau)$

$\boldsymbol{E}_i[v_2(t)] = \boldsymbol{V}'_{2,i}(t) - 14\boldsymbol{V}_{1,i}(t) - \frac{53}{5}\boldsymbol{V}_{2,i}(t) + \boldsymbol{V}_{1,i}(t)\boldsymbol{V}_{3,i}(t)$

$\boldsymbol{E}_i[v_3(t)] = \boldsymbol{V}'_{3,i}(t) - \boldsymbol{V}^2_{1,i}(t) + \frac{14}{5}\boldsymbol{V}_{3,i}(t),$ (32)

where $\boldsymbol{E}_i[v_1(t)], \boldsymbol{E}_i[v_2(t)]$ and $\boldsymbol{E}_i[v_3(t)]$ are respectively the ith-order ($i = 3, 4, 5$) residual errors functions of $v_1(t), v_2(t)$ and $v_3(t)$ while $\boldsymbol{V}_{1,i}(t), \boldsymbol{V}_{2,i}(t)$ and $\boldsymbol{V}_{3,i}(t)$ are the ith-order approximate solution of $v_1(t), v_2(t)$ and $v_3(t)$ respectively. Also, as mentioned by Kehlet and Logg (2010), the original Lorenz system can be calculated for a long interval of time, that is a small increment in t lead to a fairly long time. As a result of this, the residual error function is defined over a short interval of time. Therefore, Figures 1 below gives a Residual Error Function of $v_1(t)$.

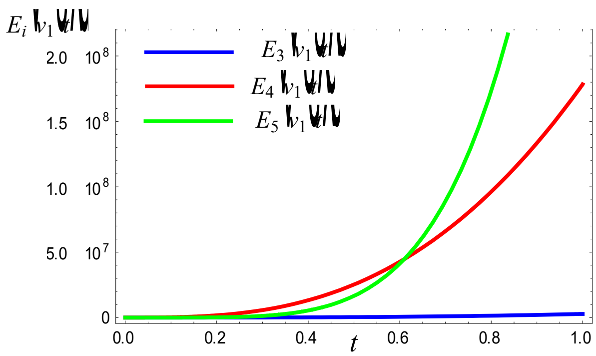

Figure 1: Residual Error Function of $v_1(t)$.

Also Figure 2 below presents a Residual Error Function of $v_2(t)$.

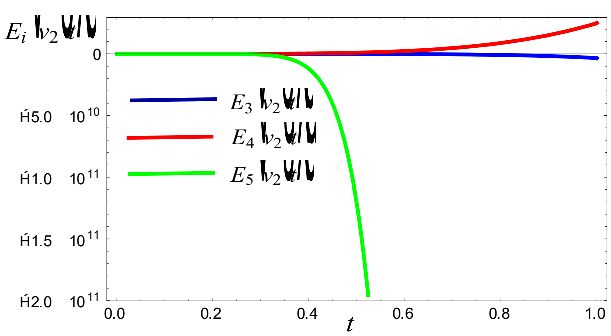

Figure 2: Residual Error Function of $v_2(t)$.

The following figure gives Residual Error Function of $v_3(t)$.

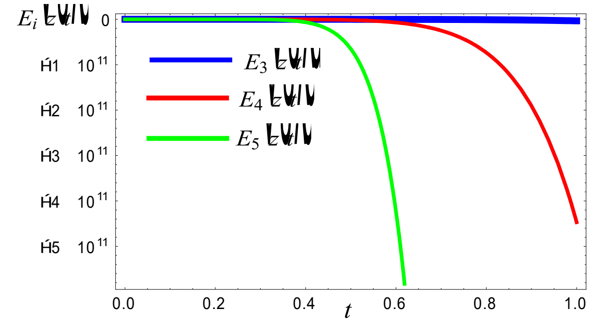

Figure 3: Residual Error Function of $v_3(t)$.

The results from Figures 1 to 3 revealed that more precise solutions could be obtained from three iterations of the proposed technique. Therefore, taking additional terms is no longer needed. Hemce, based on their respective residual error functions, there is no significant improvement in the accuracy of the approximate solution.

Spread of Infectious Diseases Model

Infectious diseases can be transferred among individuals after proto-microorganisms and parasites infect human beings (Herbert, 2000 and Sebastian *et.al.* 2010). These diseases can cause the death of a human due to their Infectivity, epidemic, and uncertainty in nature. Mathematical models play a vital role in determining the spread mechanism of these diseases. Therefore, a lot of mathematical models have been developed to describe the spread of infectious diseases. Some of these models are established to study the general laws of such diseases, and others are designed to study a particular type of the diseases (Mohammad and Abba, 2012, Meng *et.al.* 2014, Gengxin and Sheng, 2017, Ofosuhene and Azina, 2014).

Based on Kermack and McKendrick model (Wang and Wu, 2010), a widely accepted model inthe form of diffierential equations is established in by (Gengxin and Sheng, 2017).

According to this model, the number of new infections per unit time is proportional to the product of the fraction of the susceptible study population, the infected individuals and the fractional immunized and all the constant of proportionality are assumed to be equal to one (Pedro *et.al.* 2017). Similarly, the immunized population is also assumed to be susceptible after a fixed period of time τ_1. As introduced the incubation period τ_2 this model is obtained as follows:

$$v_1'(t) = v_2(t-\tau_2) - v(t)v_2(t-\tau_1)$$
$$v_2'(t) = v_1(t)v_2(t-\tau_1) - v_2(t)$$
$$v_3'(t) = v_2(t) - v_2(t-\tau_2)\,, \quad (33)$$

with initial condition

$$v_1(t) = 0,\ \ v_2(t) = 1,\ \ v_3(t) = 5,$$
(34)

Here, the fraction of susceptible study population is denoted as $v_1(t)$, $v_2(t)$ is the number of the infected individuals, and $v_3(t)$ represents the fraction immunized. To obtain the solution of this problem using the proposed method the Natural transform is applied first to both side of the Equation (33) and then simplify further using Equation (34) to obtain

$$\boldsymbol{N}^+[v_1(t)] - \frac{u}{s}\boldsymbol{N}^+[v_2(t-\tau_2) - v_1(t)v_2(t-\tau_1)] = 0$$
$$\boldsymbol{N}^+[v_2(t)] - \frac{1}{s} - \frac{u}{s}\boldsymbol{N}^+[v_1(t)v_2(t-\tau_1) - v_2(t)] = 0$$
(35)
$$\frac{u}{s}\boldsymbol{N}^+[v_3(t)] - \frac{5}{s} - \frac{u}{s}\boldsymbol{N}^+[v_2(t) - v_2(t-\tau_2)\,] = 0$$

Based on Equation (35) a nonlinear operator is defined as:

$$\mathrm{N}_1[\varphi_1(t,q)] = \mathbb{N}^+[\varphi_1(t,q)] - \frac{u}{s}\mathbb{N}^+\big[\varphi_2(t-\tau_{2,}q) - \varphi_1(t,q)\varphi_2(t-\tau_1,q)\big] = 0$$
$$\mathrm{N}_2[\varphi_2(t,q)] = \mathbb{N}^+[\varphi_2(t,q)] - \frac{1}{s} - \frac{u}{s}\mathbb{N}^+[\varphi_1(t,q)\varphi_2(t-\tau_1,q) - \varphi_2(t,q)] = 0$$
(36)
$$\mathrm{N}_3[\varphi_3(t,q)] = \mathbb{N}^+[\varphi_3(t,q)] - \frac{5}{s} - \frac{u}{s}\big[\varphi_2(t,q) - \varphi_2\big(t-\tau_{2,},q\big)\,\big] = 0,$$

Based on Equation (15) the recursive relation of the model is obtained as

$$v_{1,m}(t) = (\chi_m + \hbar_1)v_{1,m-1}(t) - \hbar_1\mathbb{N}^-\left\{\frac{u}{s}\mathbb{N}^+\big[R_1\big(v_{2,m-1}(t)\big) - H_{1m-1}\big(v_{\lambda_1}, v_{\lambda_2}, \ldots, v_{\lambda_N}\big)\big]\right\},\ \ m \geq 1$$

$$v_{2,m}(t) = (\chi_m + \hbar_2)v_{2,m-1}(t) - \hbar_2(1-\chi_m)\mathbb{N}^-\left\{\frac{1}{s}\right\} + \hbar_2\mathbb{N}^-\left\{\frac{u}{s}\mathbb{N}^+\big[R_2\big(v_{2,m-1}(t)\big) - H_{2,m-1}\big(v_{1,0}, v_{2,0}, \ldots, v_{1,m-1}, v_{2,m-1}\big)\big]\right\}$$

$$v_{3,m}(t) = (\chi_m + \hbar_3)v_{3,m-1}(t) - \hbar_3(1-\chi_m)\mathbb{N}^-\left\{\frac{5}{s}\right\} - \hbar_3\mathbb{N}^-\left\{\frac{u}{s}\mathbb{N}^+\big[R_3\big(v_{2,m-1}(t)\big)\big]\right\}, m \geq 1.$$

(37)

In this model the nonlinear functions $F_1 = F_2 = v_1(t)v_2(t-\tau_1)$ therefore, these functions can be computed using the modified He's polynomials $H_{1,m} = H_{2,m} = H_m$ defined as

$$H_{m-1}\Big(\big(v_{1,0}, v_{2,0}\big), \ldots, \big(v_{1,m-1}, v_{2,m-1}\big)\Big) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}F\left(\sum_{p=0}^{m-1} q^p\big(v_{1,p}(t), v_{2,p}(t)\big)\right)|_{q=0}$$
$$= \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}F\left[\left(\sum_{p=0}^{m-1} q^p\big(v_{1,p}(t)\big)\right)\left(\sum_{p=0}^{m-1} q^p\big(v_{2,p}(t-\tau_1)\big)\right)\right]|_{q=0} \quad (38)$$

From the given initial condition the initial approximation can be choose as

$$v_{1,0}(t) = t,\ \ \ v_{2,0}(t) = 1,\ \ \ v_{3,0}(t) = 5$$
(39)

So, using Equation (37) the components of $v_{1,m}(t), v_{2,m}(t)$ and $v_{3,m}(t)$ is obtained as:

$$v_{11}(t) = \hbar_1 v_{10}(t) - \hbar_1\mathbb{N}^-\left\{\frac{u}{s}\mathbb{N}^+\big[R_1\big(v_{2,0}(t)\big)\big]\right\}$$

$- \hbar_1 \mathbb{N}^- \left\{\frac{u}{s}\mathbb{N}^+\left[-H_0\left(v_{1,0}, v_{2,0}\right)\right]\right\} = \frac{1}{2}\hbar_1 t^2$

$v_{2,1}(t) = \hbar_2 v_{2,0}(t) - \hbar_2 \mathbb{N}^- \left\{\frac{1}{s}\right\} +$
$\hbar_2 \mathbb{N}^- \left\{\frac{u}{s}\mathbb{N}^+\left[R_2\left(v_{2,0}(t)\right) - H_0\left(v_{1,0}, v_{2,0}\right)\right]\right\} =$
$\hbar_2 t - \frac{1}{2}\hbar_2 t^2$

$v_{3,1}(t) = v_{3,0}(t) - \hbar_3 \mathbb{N}^- \left\{\frac{5}{s}\right\}$

$- \hbar_3 \mathbb{N}^- \left\{\frac{u}{s}\mathbb{N}^+\left[R_3\left(v_{2,0}(t)\right)\right]\right\} = 0$ (40)

$v_{12}(t) = \frac{1}{2}\left(\tau_2^2 + 2\tau_2\right)\hbar_1\hbar_2 - \frac{1}{4}\left[\left(\tau_1^2 + 2\tau_1 + 2\tau_2 + 2\right)\hbar_1\hbar_2 - \hbar_1 - {\hbar_1}^2\right]t^2 + \frac{1}{6}\left[(2\tau_1 + 3)\hbar_1\hbar_2 + {\hbar_1}^2\right]t^3 - \frac{1}{8}\hbar_1\hbar_2 t^4$

The remaining components is obtained as

$v_{2,2}(t) = (1 + \hbar_2)v_{2,1}(t)$

$+ \hbar_2 \mathbb{N}^- \left\{\frac{u}{s}\mathbb{N}^+\left[R_2\left(v_{2,1}(t)\right) - H_1\left(v_{1,1}, v_{2,1}\right)\right]\right\}$

$= \left(\hbar_2^2 + \hbar_2\right)t + \frac{1}{4}\left[\left(\tau_1^2 + 2\tau_2\right)\hbar_2^2 - 2\hbar_1\right]t^2$

$- \frac{1}{6}\left[(2\tau_1 + 3)\hbar_2^2 + \hbar_1\hbar_2\right]t^3 + \frac{1}{8}\hbar_2^2 t^4$

$v_{3,2}(t) = (1 + \hbar_3)v_{3,1}(t)$

$- \hbar_3 \mathbb{N}^- \left\{\frac{u}{s}\mathbb{N}^+\left[R_3\left(v_{2,1}(t)\right)\right]\right\}$

$= -\frac{1}{2}\left(\tau_2^2 + 2\tau_2\right)\hbar_2\hbar_3 t + \frac{1}{2}\hbar_2\hbar_3\tau_2 t^2$

$v_{13}(t) = (1 + \hbar_1)v_{12}(t)$

$- \hbar_1 \mathbb{N}^- \left\{\frac{u}{s}\mathbb{N}^+\left[R_1\left(v_{2,2}(t)\right) - H_2\left(v_{1,2}, v_{2,2}\right)\right]\right\}$

$= \frac{1}{48}\hbar_1\hbar_2^2 t^6 - \frac{1}{120}\left[(5\tau_1 + 6)\hbar_1\hbar_2^2 + 13\hbar_1^2\hbar_2\right]t^5$

$+ \frac{1}{8}\left[\left(24\tau_1^2 + 31\tau_1 + 6\tau_1 6\right)\hbar_1\hbar_2^2 - (24\tau_1 + 6)\hbar_1^2\hbar_2 - 12\hbar_1\hbar_2 + 2\hbar_3^2\right]t^4$

$+ \frac{1}{12}\left[(8\tau_1 + 12)\hbar_1\hbar_2 - (2\tau_1 - 3)\hbar_1^2 - \left(4\tau_1^2 - 2\tau_1 + 4\tau_2 + 4\right)\hbar_1^2\hbar_2 - \left(8\tau_1^2 + 12\tau_1^2 + 4\tau_1^2 + 9\tau_2 + 4\tau_1\tau_2 - 4\right)\hbar_1^2\hbar_2 + 3\hbar_1^3\right]t^3$

$+ \frac{1}{48}\left[\frac{1}{24}\left(4\tau_1^3 - 288\tau_1^2 + 300\tau_2^2 - 522\tau_2 - 576\right)\hbar_1^2\hbar_2 + \left(17\tau_1^4 + 30\tau_1^3 + 24\tau_2^4 + 72\tau_2^2 - 24\tau_1 + 24\tau_1\tau_2 - 24\right)\hbar_1\hbar_2^2 - \left(24\tau_1^2 + 48\tau_1 + 48\tau_2 + 48\right)\hbar_1\hbar_2 + 12\hbar_1^3 + 24\hbar_1^2\right]t^2$

$- \frac{1}{24}\left[\left(9\tau_2^4 + 30\tau_2^3 + 8\tau_1\tau_2^3 - 24\tau_2\right)\hbar_1\hbar_2^2 + \left(24\tau_2^2 + 48\tau_2\right)\hbar_1\hbar_2 - \left(8\tau_2^2 + 24\tau_2\right)\right]t$

$v_{2,3}(t) = (1 + \hbar_2)v_{2,2}(t)$

$+ \hbar_2 \mathbb{N}^- \left\{\frac{u}{s}\mathbb{N}^+\left[R_2\left(v_{2,2}(t)\right) - H_1\left(v_{1,2}, v_{2,2}\right)\right]\right\}$

$= -\frac{1}{48}\tau_2^3 t^6 + \frac{1}{120}\left[(5\tau_1 + 6)\hbar_2^3 + 13\hbar_1\hbar_2^2\right]t^5 - \frac{1}{48}\left[\left(24\tau_1^2 + 13\tau_1\right)\hbar_2^3 + (4\tau_1 + 12)\hbar_1\hbar_2^2 - 12\hbar_2^2 + 2\hbar_1^2\hbar_2\right]t^4 + \frac{1}{12}\left[\left(8\tau_1^3 + 13\tau_1^2 - 4\tau_1 - 10\right)\hbar_2^3 - (8\tau_1 + 12)\hbar_2^2 + \left(4\tau_1^2 + 2\tau_1 + 2\tau_2\right)\hbar_1\hbar_2^2 + (2\tau_1 - 3)\hbar_1\hbar_2 - \hbar_1^2\hbar_2\right]t^3 - \frac{1}{48}\left[\left(17\tau_1^4 + 30\tau_1^3 - 12\tau_1^2 + 48\tau_1 - 24\right)\hbar_2^3 - \left(24\tau_1^2 + 48\tau_1\right)\hbar_2^2 + \frac{1}{24}\left(4\tau_1^3 + 12\tau_2^2 + 24\tau_2\right)\hbar_1\hbar_2^2 + 24\hbar_2\right]t^2 + \left[\hbar_2^3 + 2\hbar_2^2 + \hbar_2\right]t$

(41)

$v_{3,3}(t) = (1 + \hbar_3)v_{3,2}(t) - \hbar_3 \mathbb{N}^- \left\{\frac{u}{s}\mathbb{N}^+\left[R_3\left(v_{2,2}(t)\right)\right]\right\}$

$= \frac{1}{8}\hbar_2^2\hbar_3\tau_1 t^4 + \frac{1}{12}\left[\left(4\tau_2^2 - \tau_1^2 + 9\tau_2 + 4\tau_1\tau_2\right)\hbar_2^2\hbar_3 + 2\tau_2\hbar_1\hbar_2\hbar_3\right]t^3 - \frac{1}{4}\left[\left(2\tau_2^3 + 6\tau_2 + 2\tau_1\tau_2\right)\hbar_2^2\hbar_3 - 2\tau_2\hbar_2\hbar_3^2 - 4\tau_2\hbar_2\hbar_3 + \tau_2^2\hbar_1\hbar_2\hbar_3\right]t^2 - \frac{1}{24}\left[\left(9\tau_2^4 + 30\tau_2^3 + 8\tau_1\tau_2^3 - 24\tau_2\right)\hbar_2^2\hbar_3 + \left(12\tau_2^2 - 24\tau_2\right)\hbar_2\hbar_3^2 + 4\tau_2^3\hbar_1\hbar_2\hbar_3\right]t.$

Bima Journal of Science and Technology, Vol. 7 (1) Mar, 2023 ISSN: 2536-6041

DOI: 0.56892/bima.v7i01.393

Since this model has no exact solution therefore, the residual error function is also adopted to measure the accuracy of this approximation which defined in the following equation.

$$\boldsymbol{E}[v_1(t)] = \boldsymbol{V}_1^{'}(t) - v_2(t-\tau_2) + v_1(t)v_2(t-\tau_1)$$
$$\boldsymbol{E}[v_2(t)] = \boldsymbol{V}_2^{'}(t) - v_1(t)v_2(t-\tau_1) + v_2(t)$$
$$\boldsymbol{E}[v_3(t)] = \boldsymbol{V}_3^{'}(t) - v_2(t) + v_2(t-\tau_2) \quad , \quad (42)$$

where $\boldsymbol{E}[v_1(t)]$, $\boldsymbol{E}[v_2(t)]$ and $\boldsymbol{E}[v_3(t)]$ are respectively the residual error functions of $v_1(t), v_2(t)$. And $v_3(t)$. While $\boldsymbol{V}1(t), \boldsymbol{V}2(t)$ and $\boldsymbol{V}3(t)$ are third-order approximate solutions of $v_1(t), v_2(t)$ and $v_3(t)$ respectively.

Therefore, using the optimal values of the auxiliary parameters, $\hbar_1 = \hbar_2 = \hbar_3 =- 1$ Figures 4 describes the residual error functions of Equation (42) for the third-order approximation of this model with different values of time-delays $\tau_1 = 0.01$ and $\tau_2 = 0.02$.

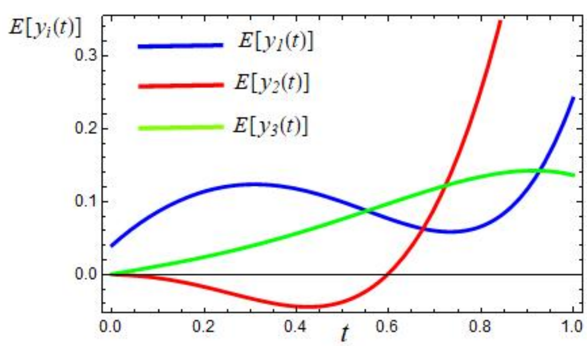

Figure 4: Residual Error Functions of Equation (42) with $\tau_1 = 0.01$ and $\tau_2 = 0.02$.

While Figures 5 below describes the residual error functions of Equation (42) for the third-order approximation of this model with different values of time-delays $\tau_1 = 0.1$ and $\tau_2 = 0.2$.

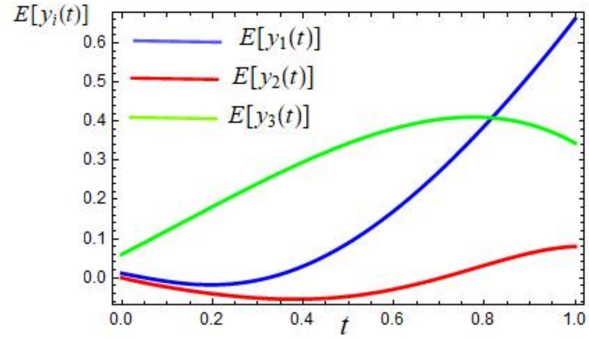

Figure 5: Residual Error Functions of Equation (42) with $\tau_1 = 0.1$ and $\tau_2 = 0.2$.

Observed from Figures 4 and 5 apart from the auxiliary parameters, the time delays τ_1 and τ_2 have indeed a significant infiuence on the global dynamics of the model. Therefore, according to these Figures of the residual error functions, the third-order approximation of this model obtained using the proposed technique attain a better result of $v_1(t)$, and $v_3(t)$ at $\tau_1 = 0.01$, $\tau_2 = 0.02$ and for $v_2(t)$ at $\tau_1 = 0.1$, $\tau_2 = 0.2$ when the auxiliary parameters $\hbar_i =- 1$.

CONCLUSION

In this work, the He's polynomial is modified to ease the computational dificulties of nonlinear terms for the two models of the system of Retarded DDEs. The solution is obtained using efficient analytical approach. The results reveal that the approach gives rise to an easy and straightforward means of solving these models analytically. Consequently by choosing an optimal value of auxiliary parameters the more precise approximate solutions are obtained from only three iterations number of terms. Some figures are used to demonstrate the accuracy of the third-order approximations based on the residual error functions. Therefore, the method has indeed a general meaning which can also be applied to solve various models of nonlinear DDEs.

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