ON THE GALOIS THEORY OF DIFFERENTIAL EQUATIONS

[1*]BABAYO M. A AND [2]MUHAMMAD A. B

Department of Mathematics and Computer Science, Faculty of Science, Federal University of Kashere, P. M. B. 0182, Nigeria.

Department of Mathematics, Faculty of Science, Gombe State University, Gombe, Nigeria.

Corresponding Author: baabaayo2014@gmail.com and bapdukku@gmail.com

ABSTRACT

The regular n-gon $\theta^n = 1$ may also contain another m-gon $\theta^m = 1$ inscribed in it. Since both $\theta^n = 1$ and $\theta^m = 1$ can imply and $\theta^{n+m} = 1$. Then $\frac{d^n y}{dx^n} - 1 = 0$ and $\frac{d^m y}{dx^m} - 1 = 0$ as lengths of the arcs $\theta^n = 1$ and $\theta^m = 1$ respectively, can imply $\left(\frac{d^n y}{dx^n} - 1\right)\left(\frac{d^m y}{dx^m} - 1\right) = 0$. That is, $y = \frac{d^{n+m} y}{dx^{n+m}} - \frac{d^n y}{dx^n} - \frac{d^m y}{dx^m} + 1 = 0$. This article shows the generalized concept of this idea by applying / introducing the Semigroup (Galois) theory.

Keywords: Action and Reaction, Regular m-Gon, Solvable Quartics.

INTRODUCTION

Galois Theory, by Evariste Galois (1811 – 1832) was the first to introduce application of Group Theory to the solution of Algebraic equations. Basically, it provides the relationship between the structure of Groups and the structure of Fields and applies this relationship to describe how the roots of Polynomials relate to each other.

It was Newton (1687) that stated that “for every action, there is an equal and opposite reaction”, and as well introduced calculus in studying our place, i.e. our position, in this system. Haven referred this action as “a Force”, he then postulated the first law of motion such that “a body continues in its state of rest or uniform motion unless it is impacted upon by a Force”. But then a force **f** is a vector quantity, so **-f** might be a friction or any form of inpediment. $Hence \{\mathbf{f},\ \mathbf{-f},\ \mathbf{e}\}$ is a group. It seemed as if Newton had the idea of grouping in his mind. This might be possible because of his attempt towards theory of everything. He also believed that there is nothing in the universe that is not governable by equations. This idea was backed by Stephen (2015) who said that the universe is an internal direct product of symmetry groups.

The solvability of equations or their non-solvability was the question of Abel (1824) which Galois (1864) solved using the introduction of group theory. For example, $y = mx + c$ is solvable since, $f\left(\frac{-c}{m}\right) = 0$.

The quadratic equations have the mighty formula by Shubtra (1879); even mightier than quadratic equations, because it solves equations of the form $y = ax^4 + bx^2 + c = 0$. The cubics are solvable by Tartaglia’s (1964) formula. The non-solvability of the quintics was because, as was asserted by Heinsten (1964), of $A_5 -$ since alternating group of length 5 – is not solvable. This is related to saying five dimensional graph has no origin; i.e. no five perpendicular lines meeting at origin. Since the Galois Theory provides the relationship between the structure of Groups and the structure of Fields and applies this relationship to describe how the roots of Polynomials relate to each other, the task is to extend this to Calculus. This study therefore, is intended to further generalize the use of the Semigroup Theory in determining the solvability or otherwise of

Differential Equations as applied by Galois (1864) using, and or, introduction of Group Theory.

MATERIALS AND METHODS

Consider the set $S = \{x, a, b, c, d, e\}$. Then, by Russel (1919), S is a collection; but not a connection, as in the preface of Higgins (1992) by G. B. Preston.

For x to be connected to a, S should be a basement (a length) so that $\{x, a\}$is contained in a subset of the area A, as length L, times breath, S; i. e. $S \times L$, where $L = S$ in this case.

Thus, any subset of S^2 is a relation (R). Hence, x can be connected with a by xRa. An equivalence relation [10,11] is also a subset of R. If x can be connected with a by $x = a$, as well as with b by $x = b$; then, $x - a = 0$ and $x - b = 0$.

Hence, $(x - a)(x - b) = 0$. That is, $x^2 - (a + b)x + ab = 0$.

From Smith and Rowland (2002), to look at the earth in its true shape, its picture can be taken from the moon. Thus, for $y = x^2 - (a + b)x + ab$ such that: $y = f(x) = 0$; then the quantics can be expressed as $y = (x - a)^2(x - b)^2$. This implies that there are solvable quintics, for example, $y = (x - a)^3(x - b)(x - c)$, not necessary by radicals. This is because the only highest radical solution known is $y = (x - a)(x - b)(x - c)$.

Again, xRa is not the only suitable predictable connection between x^n, and a or $x = a$. Another one is $x^n R\, a$, where $a = b^n$ or so, and analogously is $(x - a)^n\, R\, 0$.

Since $y = (x - a)(x - b)(x - c)$ is solvable, then $y = (x^u - a)(x^v - b)(x^w - c)$ is also solvable. Hence, this implies that $\left(\frac{d^u y}{dx^u} - a\right)\left(\frac{d^v y}{dx^v} - b\right)\left(\frac{d^w y}{dx^w} - c\right) = 0$ is solvable.

SOLUTIONS

How is $\frac{d^u y}{dx^u} - a = 0$ solvable?

Let v be a vector. Then $v = ai + bj$, with magnitude, $\sqrt{a^2 + b^2}$.

This is saying, to have length of an arc, one needs a plane that contains the arc Let one of the vector planes be $a + ib$ or $a + bj$ or $ai + b$ or $aj + b$ or down the line, where $i = j = \sqrt{-1}$.

Let $z = a + bi$ or the down. Then, $(a, b) \subseteq V$ and $z = re^{i\theta}$;

where $e^{i\theta} = cos\theta + i\, sin\theta$, because $a = r\, cos\theta$ and $b = r\, sin\theta$, where $\overline{0a} = r$.

Hence, $z^m = r^m e^{im\theta}$, where $e^{im\theta} = \cos m\theta + i \sin m\theta$.

From Seymour, et. al.(2009), let $m = \frac{1}{n}$; then $z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{\frac{\theta}{n}i} = r^{\frac{1}{n}}(cos\frac{\theta}{n} + i\, sin\frac{\theta}{n})$.

For $\frac{d^n y}{dx^n} = 1$, then $\frac{dy}{dx} = 1^{\frac{1}{n}} e^{\frac{\theta}{n}i}$, where $\theta = 0$; which implies $\frac{dy}{dx} = 1^{\frac{0}{n}} e^{0i}$.

But, this is just one of the expected solutions of $\frac{d^n y}{dx^n} - 1 = 0$ and $\theta = 360$ is the same as $\theta = 0$ becouse $e^{360i} = cos360 + i \sin 360$ and $e^{0i} = cos0 + i \sin 0$. Applying a partition of $360 = 2\pi$ into n number of times therefore gives:

$\frac{dy}{dx} = 1^{\frac{1}{n}} e^{\frac{2\pi}{n}i}, \frac{dy}{dx} = 1^{\frac{2}{n}} e^{\frac{2\pi}{n}2i}, \frac{dy}{dx} = 1^{\frac{3}{n}} e^{\frac{2\pi}{n}3i}, \ldots, \frac{dy}{dx} = 1^{\frac{n}{n}} e^{\frac{2\pi}{n}ni}$;

$\frac{dy}{dx} = 1^{\frac{0}{n}} e^{\frac{2\pi}{n}(0)i}, \frac{dy}{dx} = 1^{\frac{1}{n}} e^{\frac{2\pi}{n}i}, \frac{dy}{dx} = 1^{\frac{2}{n}} e^{\frac{2\pi}{n}2i}, \frac{dy}{dx} = 1^{\frac{3}{n}} e^{\frac{2\pi}{n}3i}, \ldots, \frac{dy}{dx} = 1^{\frac{n}{n}} e^{\frac{2\pi}{n}ni}$. Thus,

$\frac{dy}{dx} = cos\frac{2\pi}{n} + i\, sin\frac{2\pi}{n}, \frac{dy}{dx} = cos\frac{4\pi}{n} + i\, sin\frac{4\pi}{n}$, $\frac{dy}{dx} = cos\frac{6\pi}{n} + i\, sin\frac{6\pi}{n}$,

$\frac{dy}{dx} = \quad cos\frac{8\pi}{n} + i\,sin\frac{8\pi}{n}$, ..., $\frac{dy}{dx} = \cos 2\pi + i \sin 2\pi$.

This implies:

$$\left[\frac{dy}{dx} - cos\frac{2\pi}{n} + i\,sin\frac{2\pi}{n}\right] \times \left[\frac{dy}{dx} - cos\frac{4\pi}{n} + i\,sin\frac{4\pi}{n}\right] \times \left[\frac{dy}{dx} - cos\frac{6\pi}{n} + i\,sin\frac{6\pi}{n}\right], \left[\frac{dy}{dx} - cos\frac{8\pi}{n} + i\,sin\frac{8\pi}{n}\right], \ldots, \left[\frac{dy}{dx} - \cos 2\pi + i \sin 2\pi\right] = \frac{d^n y}{dx^n} - 1\,;$$

$$\left[\frac{dy}{dx} - \sqrt[n]{a}\left(cos\frac{2\pi}{n} + i\,sin\frac{2\pi}{n}\right)\right] \times \left[\frac{dy}{dx} - \sqrt[n]{a}\left(cos\frac{4\pi}{n} + i\,sin\frac{4\pi}{n}\right)\right] \times \left[\frac{dy}{dx} - \sqrt[n]{a}\left(cos\frac{6\pi}{n} + i\,sin\frac{6\pi}{n}\right)\right],$$

$$\left[\frac{dy}{dx} - \sqrt[n]{a}\left(cos\frac{8\pi}{n} + i\,sin\frac{8\pi}{n}\right)\right], \ldots, \left[\frac{dy}{dx} - \sqrt[n]{a}(\cos 2\pi + i \sin 2\pi)\right] = \frac{d^n y}{dx^n} - a.$$

This is related to the fundamental theorem of algebra: that every group is a direct product of cyclic groups, each of prime power order [14].

RESULTS

1. Since $f(S) = \{f(x), f(a), f(b), f(c), f(d), f(e)\}$ and $f^{'}(x)R\,f(a)$;
Then $f^n(x)Rf(a)$ such as $\frac{d^n y}{dx^n} - 1 = 0$ is solvable, and so does $\frac{d^n y}{dx^n} - a = 0$;
Also $\frac{d^m y}{dx^m} - b = 0$ is also solvable, and so does $y = \left(\frac{d^n y}{dx^n} - a\right)\left(\frac{d^m y}{dx^m} - b\right) = 0$.
That is, $y = \frac{d^{n+m} y}{dx^{n+m}} - b\frac{d^n y}{dx^n} - a\frac{d^m y}{dx^m} + ab = 0$.
This has tackled $y = \left(\frac{d^n y}{dx^n} - a\right)\left(\frac{d^n y}{dx^n} - b\right) = 0$, $y = \left(\frac{d^n y}{dx^n} - a\right)\left(\frac{d^n y}{dx^n} - a\right) = 0$.
These ideas could be summed to $y = \left(\frac{d^n y}{dx^n} - a\right)\left(\frac{d^m y}{dx^m} - b\right)\left(\frac{d^u y}{dx^u} - c\right)\left(\frac{d^v y}{dx^v} - d\right)$ for solvability – related to solvability of the quartics or $y = \left(\frac{d^n y}{dx^n} - a\right)^2\left(\frac{d^m y}{dx^m} - b\right)^2$ or down the line.

2. For $\dot{f}(S) = \{f^{'}(x), f^{'}(a_1), f^{'}(a_2), f^{'}(a_3), \ldots, f^{'}(a_n), \ldots\}$, one can use:
$f^{'}(a_u) \equiv f^{'}(a_v)\,mod\,w$ if and only if $f^{'}\left(a_{\frac{u-v}{w}}\right) \in \dot{f}(S)$, where $\dot{f}(S)$ is a set of differentiable functions and f' is $\frac{d}{dx}f(x)$. Hence, $\frac{d^n y}{dx^n} - 1 = 0$ is solvable using $\left(\frac{d^a y}{dx^a} - 1\right) \equiv \left(\frac{d^b y}{dx^b} - 1\right) mod\, d$ if and only if $\left(\frac{d^{\frac{a-b}{d}} y}{dx^{\frac{a-b}{d}}} - 1\right) \in \frac{d^n y}{dx^n} - 1$.
Thus, define the canonical map $\theta: S \to {S}/{I}$ by $\theta\left(\frac{d^n y}{dx^n} - 1\right) = \frac{d^{nm} y}{dx^{nm}} - 1$.

3. The Onto-Homomorphism can be shown as follows:

(i) Homomorphism: The $\theta\left[\left(\frac{d^a y}{dx^a} = 1\right) \times \left(\frac{d^b y}{dx^b} = 1\right)\right] = \theta\left(\left(\frac{d^{a+b} y}{dx^{a+b}} - 1\right)\right)$;

and by definition of θ, it is $\left(\frac{d^{(a+b)m} y}{dx^{(a+b)m}} = 1\right)$.

That is, $\left(\frac{d^{am}y}{dx^{am}}=1\right)\times\left(\frac{d^{bm}y}{dx^{bm}}=1\right)$ which is $\theta\left(\frac{d^{a}y}{dx^{a}}=1\right)\times\theta\left(\frac{d^{b}y}{dx^{b}}=1\right)$.

(ii) Ontoness: Let $\theta^{-1}\left(\frac{d^{am}y}{dx^{am}}=1\right)=\theta^{-1}\left(\frac{d^{bm}y}{dx^{bm}}=1\right)$.

Then $\left(\frac{d^{a}y}{dx^{a}}=1\right)=\left(\frac{d^{b}y}{dx^{b}}=1\right)$.

⇨ $\left(\frac{d^{a}y}{dx^{a}}=1\right)\times\left(\frac{d^{a}y}{dx^{a}}=1\right)\times \ldots m\ times$

$=\left(\frac{d^{b}y}{dx^{b}}=1\right)\times\left(\frac{d^{b}y}{dx^{b}}=1\right)\times \ldots m\ times.$

⇨ $\left(\frac{d^{am}y}{dx^{am}}=1\right)=\left(\frac{d^{bm}y}{dx^{bm}}=1\right)$.

4. Define the canonical map $\theta: {}^{S}/_{I}\to\frac{{}^{S}/_{I}}{J}$ by $\theta\left(\frac{d^{nm}y}{dx^{nm}}-1\right)=\frac{d^{nmp}y}{dx^{nmp}}-1$.; then the onto homomorphism can be shown as follows:

(i) Homomorphism: The $\theta\left[\left(\frac{d^{am}y}{dx^{am}}=1\right)\times\left(\frac{d^{bm}y}{dx^{bm}}=1\right)\right]=\theta\left(\left(\frac{d^{(a+b)m}y}{dx^{(a+b)m}}-1\right)\right)$

and by definition of θ, it is $\left(\frac{d^{(a+b)mp}y}{dx^{(a+b)mp}}=1\right)$;

⇨ $\left(\frac{d^{amp}y}{dx^{amp}}=1\right)\times\left(\frac{d^{bmp}y}{dx^{bmp}}=1\right)$; which is $\theta\left(\frac{d^{am}y}{dx^{am}}=1\right)\times\theta\left(\frac{d^{bm}y}{dx^{bm}}=1\right)$.

(ii) For Ontoness:

Let $\theta^{-1}\left(\frac{d^{amp}y}{dx^{amp}}=1\right)=\theta^{-1}\left(\frac{d^{bmp}y}{dx^{bmp}}=1\right)$. Then $\left(\frac{d^{am}y}{dx^{am}}=1\right)=\left(\frac{d^{bm}y}{dx^{bm}}=1\right)$. ⇨ $\left(\frac{d^{am}y}{dx^{am}}=1\right)\times\left(\frac{d^{am}y}{dx^{am}}=1\right)\times \ldots p\ times$

$=\left(\frac{d^{bm}y}{dx^{bm}}=1\right)\times\left(\frac{d^{bm}y}{dx^{bm}}=1\right)\times \ldots p\ times.$

⇨ $\left(\frac{d^{amp}y}{dx^{amp}}=1\right)=\left(\frac{d^{bmp}y}{dx^{bmp}}=1\right)$.

CONCLUSION

Thus it can be concluded that, if $S=\{g^x,g^a,g^b,g^c,g^d,g^e\}$, where $g^{e+?}=g^x$, then S is a group an example of which is $\theta^n=1$;

That is, $[\theta^n=1]=\{\theta^{n-1}=1,\theta^{n-2}=1,\theta^{n-3}=1,\theta^{n-4}=1,\ldots,\theta^{n-n}=1\}$. Thus, $\theta^{n-u}\equiv\theta^{n-v} mod\ w$ if and only if $\theta^{\frac{(n-u)-(n-v)}{w}}\in[\theta^n=1]$.

But, a naturally existing semigroup is $\theta^n=\theta$. That is, $\frac{d^ny}{dx^n}-\frac{dy}{dx}=0$.

Hence, $\frac{dy}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}-1\right)=0$. This implies y is a constant: $\int 0\,dx=0+c$, where c is a constant. This constant could be must likely be 0.

Another Semigroup is $\left[\frac{d^{\infty}y}{dx^{\infty}}-1\right]=\left\{1,\frac{dy}{dx}-1,\frac{d^2y}{dx^2}-1\ldots,\frac{d^ny}{dx^n}-1,\ldots\right\}$.

In this case, one can apply: $\left(\frac{d^{a}y}{dx^{a}}-1\right)\equiv\left(\frac{d^{b}y}{dx^{b}}-1\right) mod\ d$ if and only if

$\left(\frac{d^{\frac{a-b}{d}}y}{dx^{\frac{a-b}{d}}}=1\right)\in\left[\frac{d^ny}{dx^n}-1\right]$.

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