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SEMIGROUP SOLVABILITY OF SURFACE INTEGRALS

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ABSTRACT

The study, algebraicly explicates the general solvability of the Surface Integral $\iint_S f(x)dxdy$; by considering $\lim_{\Delta x_i \Delta y_i \to 0} \sum f(\xi_i)\Delta x_i \Delta y_i = \iint_S f(x)dxdy$. Then $\left|\sum f(\xi_i)\Delta x_i \Delta y_i - \iint_S f(x)dxdy\right| < \varepsilon$ whenever $|\Delta x_i \Delta y_i| < \delta$, where $f(\xi_i) \in \Delta x_i \Delta y_i$ and ε and δ are infinitesimally small numbers.

Keywords: Composition Series; Decomposition; Semigroup; Solvability; Surface Integrals

INTRODUCTION

It is a known fact that Calculus was introduced by Newton (1687) in order to study the length of some arcs $\frac{dy}{dx}$ and the areas under the curves $\int_C f(x)dx$. For example, $\frac{d}{dr}(\pi r^2) = 2\pi r$ and $\int (2\pi r)dr = \pi r^2$.

The Semigroup, which was first introduced by Suschkewitz (1928), intrudes, since $2\pi r_1 + 2\pi r_2$ is $2\pi(r_1 + r_2)$.

Let $r_1 + r_2 = r_3$. Then $2\pi r_3$ is a circle of radius r_3.
Thus, the closure property of semigroups is established.
The associativity is: $2\pi\big(r_1 + (r_2 + r_3)\big) = 2\pi\big((r_1 + r_2) + r_3\big)$.

Hence, the unit circle, $\theta^n = 1$ found in the scripts of [3,4,5] could be set such that $n = r_1 \times r_2 \times r_3 \ldots \times r_{m-1} \times r_m$ (a prime factor decomposition [6,7,8]).

An example is $\theta^n = 1$ and $\theta^m = 1$ implies $\theta^{n+m} = 1$ and $\theta^{u+(v+w)} = \theta^{(u+v)+w} = 1$.

This study is intended to show the introduction of Semigroup concepts to the determination of solvability of Surface Integral, under some specified domain.

MATERIALS AND METHODS

Let $\varepsilon, \delta \in (0,1) \subseteq \mathbb{R}$. That is, ε and δ are small numbers between 0 and 1 on the real-line;

Then, given $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$\left|\sum f(\xi_i)\Delta x_i \Delta y_i - \iint_S f(x)dxdy\right| < \varepsilon; \text{ whenever } |\Delta x_i \Delta y_i| < \delta;$$

where $f(\xi_i) \in \Delta x_i \Delta y_i$.

Then $\iint_S f(x)dxdy$ is a surface integral, where $\Delta x_i = (x - x_i)$;
and $\Delta y_i = y - y_i$.

The closure is from triangular inequality $(|A| + |B| > |A + B|)$;
for all cases of A, B as real numbers: $A, B > 0$, $A, B < 0$ or $A > 0$ and $B < 0$. Thus,

$$\left(\left|\sum f(\xi_i)\Delta x_i\Delta y_i - \iint_{S_i} f(x)dxdy\right| < \varepsilon\right) + \left(\left|\sum f(\xi_j)\Delta x_j\Delta y_j - \iint_{S_j} f(x)dxdy\right| < \varepsilon\right) = \left(\left|\sum f(\xi_{i+j})\Delta x_{i+j}\Delta y_{i+j} - \iint_{S_{i+j}} f(x)dxdy\right| < \varepsilon\right);$$

whenever $\left|\Delta x_{i+j}\Delta y_{i+j}\right| < \delta$

and the associativity estalished from $f(\xi_{i+(j+k)}) = f(\xi_{(i+j)+k})$.

Define: $$\left(\left|\sum f(\xi_a)\Delta x_a\Delta y_a - \iint_{S_a} f(x)dxdy\right| < \varepsilon\right) \equiv \left(\left|\sum f(\xi_b)\Delta x_b\Delta y_b - \iint_{S_b} f(x)dxdy\right| < \varepsilon\right) \text{mod d, if and only if:}$$

$$\left(\left|\sum f\left(\xi_{\frac{a-b}{d}}\right)\Delta x_{\frac{a-b}{d}}\Delta y_{\frac{a-b}{d}} - \iint_{S_{\frac{a-b}{d}}} f(x)dxdy\right| < \varepsilon\right) \in \left(\left|\sum f(\xi_i)\Delta x_i\Delta y_i - \iint_{S_i} f(x)dxdy\right| < \varepsilon\right).$$

Then, this congruence definition is an equivalence relation.

Hence, one can define a naturally existing canonical map by:

$$\theta\left(\left|\sum f(\xi_a)\Delta x_a\Delta y_a - \iint_{S_a} f(x)dxdy\right| < \varepsilon\right) = \left(\left|\sum f(\xi_{a+b})\Delta x_{a+b}\Delta y_{a+b} - \iint_{S_{a+b}} f(x)dxdy\right| < \varepsilon\right);$$

Thus, establishing the Onto-Homomorphism.

SOLUTION AND RESULTS

(i) It can be noticed that:

$$\left|\sum f(\xi_{a+b})\Delta x_{a+b}\Delta y_{a+b} - \iint_{S_{a+b}} f(x)dxdy\right| < \varepsilon$$

is an ideal of $\left|\sum f(\xi_a)\Delta x_a\Delta y_a - \iint_{S_a} f(x)dxdy\right| < \varepsilon$,

because: $$\left(\left|\sum f(\xi_a)\Delta x_a\Delta y_a - \iint_{S_a} f(x)dxdy\right| < \varepsilon\right) + \left(\left|\sum f(\xi_{a+b})\Delta x_{a+b}\Delta y_{a+b} - \iint_{S_{a+b}} f(x)dxdy\right| < \varepsilon\right) = \left(\left|\sum f(\xi_{2a+b})\Delta x_{2a+b}\Delta y_{2a+b} - \iint_{S_{2a+b}} f(x)dxdy\right| < \varepsilon\right) \subseteq \left|\sum f(\xi_{a+b})\Delta x_{a+b}\Delta y_{a+b} - \iint_{S_{a+b}} f(x)dxdy\right| < \varepsilon;$$

Also :

$$\text{L}\left(\left|\sum f(\xi_{a+b})\Delta x_{a+b}\Delta y_{a+b} - \iint_{S_{a+b}} f(x)dxdy\right| < \varepsilon\right) + \left(\left|\sum f(\xi_a)\Delta x_a\Delta y_a - \iint_{S_a} f(x)dxdy\right| < \varepsilon\right) = \left(\left|\sum f(\xi_{2a+b})\Delta x_{2a+b}\Delta y_{2a+b} - \iint_{S_{2a+b}} f(x)dxdy\right| < \varepsilon\right) \subseteq \left|\sum f(\xi_{a+b})\Delta x_{a+b}\Delta y_{a+b} - \iint_{S_{a+b}} f(x)dxdy\right| < \varepsilon\ .$$

This, defines the naturally existing canonical map $\theta: S \to {S}/{I}$ by:

$$\theta\left(\left|\sum f(\xi_a)\Delta x_a\Delta y_a - \iint_{S_a} f(x)dxdy\right| < \varepsilon\right)$$

$$= \left(\left|\sum f(\xi_{a+b})\Delta x_{a+b}\Delta y_{a+b} - \iint_{S_{a+b}} f(x)dxdy\right| < \varepsilon\right).$$

Then the Onto Homomorphism is established as follows:

The preservation of composition is:

$\theta\begin{bmatrix}\left(\left|\sum f(\xi_u)\Delta x_u\Delta y_u - \iint_{S_u} f(x)dxdy\right| < \varepsilon\right)\\ + \left(\left|\sum f(\xi_v)\Delta x_v\Delta y_v - \iint_{S_v} f(x)dxdy\right| < \varepsilon\right)\end{bmatrix}$ is by triangular inequality, $\theta\left(\left|\sum f(\xi_{u+v})\Delta x_{u+v}\Delta y_{u+v} - \iint_{S_{u+v}} f(x)dxdy\right| < \varepsilon\right)$ which by definition of θ is: $\left(\left|\sum f\left(\xi_{(u+v)+b}\right)\Delta x_{(u+v)+b}\Delta y_{(u+v)+b} - \iint_{S_{(u+v)+b}} f(x)dxdy\right| < \varepsilon\right)$ which is:

$$\left(\left|\sum f(\xi_{u+b})\Delta x_{u+b}\Delta y_{u+b} - \iint_{S_{u+b}} f(x)dxdy\right| < \varepsilon\right) + \left(\left|\sum f(\xi_{v+b})\Delta x_{v+b}\Delta y_{v+b} - \iint_{S_{v+b}} f(x)dxdy\right| < \varepsilon\right)$$

which is:

$$\theta\left(\left|\sum f(\xi_u)\Delta x_u\Delta y_u - \iint_{S_u} f(x)dxdy\right| < \varepsilon\right) + \theta\left(\left|\sum f(\xi_v)\Delta x_v\Delta y_v - \iint_{S_v} f(x)dxdy\right| < \varepsilon\right),$$

as required.

For the Onto-ness:

Let $\theta^{-1}\left(\left|\sum f(\xi_{u+b})\Delta x_{u+b}\Delta y_{u+b} - \iint_{S_{u+b}} f(x)dxdy\right| < \varepsilon\right)$;

Be equal to $\theta^{-1}\left(\left|\sum f(\xi_{v+b})\Delta x_{v+b}\Delta y_{v+b} - \iint_{S_{v+b}} f(x)dxdy\right| < \varepsilon\right)$.

Then, by definition:

$$\left(\left|\sum f(\xi_u)\Delta x_u\Delta y_u - \iint_{S_u} f(x)dxdy\right| < \varepsilon\right) = \left(\left|\sum f(\xi_v)\Delta x_v\Delta y_v - \iint_{S_v} f(x)dxdy\right| < \varepsilon\right).$$

That is to say:

$$\left(\left|\sum f(\xi_u)\Delta x_u\Delta y_u - \iint_{S_u} f(x)dxdy\right| < \varepsilon\right) + \left(\left|\sum f(\xi_b)\Delta x_b\Delta y_b - \iint_{S_b} f(x)dxdy\right| < \varepsilon\right)$$
$$= \left(\left|\sum f(\xi_v)\Delta x_v\Delta y_v - \iint_{S_v} f(x)dxdy\right| < \varepsilon\right) + \left(\left|\sum f(\xi_b)\Delta x_b\Delta y_b - \iint_{S_b} f(x)dxdy\right| < \varepsilon\right).$$

Which implies:

$$\left(\left|\sum f(\xi_{u+b})\Delta x_{u+b}\Delta y_{u+b} - \iint_{S_{u+b}} f(x)dxdy\right| < \varepsilon\right) = \left(\left|\sum f(\xi_{v+b})\Delta x_{v+b}\Delta y_{v+b} - \iint_{S_{v+b}} f(x)dxdy\right| < \varepsilon\right)..$$

(ii) Also naturally existing canonical map $\theta: S/_I \to \frac{S/_I}{J}$ can be defined by: $\theta\left(\left|\sum f(\xi_{a+b})\Delta x_{a+b}\Delta y_{a+b} - \iint_{S_{a+b}} f(x)dxdy\right| < \varepsilon\right)$

$= \left(\left|\sum f(\xi_{a+b+c})\Delta x_{a+b+c}\Delta y_{a+b+c} - \iint_{S_{a+b+c}} f(x)dxdy\right| < \varepsilon\right)$. Then, the Onto-Homomorphism can be shown as follows:

The preservation of composition is:

$$\theta\left[\left(\left|\sum f(\xi_{u+a})\Delta x_{u+a}\Delta y_{u+a} - \iint_{S_{u+a}} f(x)dxdy\right| < \varepsilon\right) + \left(\left|\sum f(\xi_{v+b})\Delta x_{v+b}\Delta y_{v+b} - \iint_{S_{v+b}} f(x)dxdy\right| < \varepsilon\right)\right].$$

(iii) The triangular inequality:

$$\theta\left(\left|\sum f\big(\xi_{(u+v)+(a+b)}\big)\Delta x_{(u+v)+(a+b)}\Delta y_{(u+v)+(a+b)} - \iint_{S_{(u+v)+(a+b)}} f(x)dxdy\right| < \varepsilon\right),$$

Which by definition of θ is: $\left(\left|\sum f\big(\xi_{(u+v)+(a+b)+c}\big)\Delta x_{(u+v)+(a+b)+c}\Delta y_{(u+v)+(a+b)+c} - \iint_{S_{(u+v)+(a+b)+c}} f(x)dxdy\right| < \varepsilon\right)$;

Which implies:

$$\left(\left|\sum f(\xi_{u+a+c})\Delta x_{u+a+c}\Delta y_{u+a+c} - \iint_{S_{u+a+c}} f(x)dxdy\right| < \varepsilon\right) + \left(\left|\sum f(\xi_{v+b+c})\Delta x_{v+b+c}\Delta y_{v+b+c} - \iint_{S_{v+b+c}} f(x)dxdy\right| < \varepsilon\right)$$

which is: $\theta\left(\left|\sum f(\xi_{u+a})\Delta x_{u+a}\Delta y_{u+a} - \iint_{S_{u+a}} f(x)dxdy\right| < \varepsilon\right)$

$= \theta\left(\left|\sum f(\xi_{v+b})\Delta x_{v+b}\Delta y_{v+b} - \iint_{S_{v+b}} f(x)dxdy\right| < \varepsilon\right)$, as required.

For Onto-ness: Let

$$\theta^{-1}\left(\left|\sum f(\xi_{u+a+c})\Delta x_{u+a+c}\Delta y_{u+a+c} - \iint_{S_{u+a+c}} f(x)dxdy\right| < \varepsilon\right) = \theta^{-1}\left(\left|\sum f(\xi_{v+b+c})\Delta x_{v+b+c}\Delta y_{v+b+c} - \iint_{S_{v+b+c}} f(x)dxdy\right| < \varepsilon\right).$$

Then, by definition:

$$\left(\left|\sum f(\xi_{u+a})\Delta x_{u+a}\Delta y_{u+a} - \iint_{S_{u+a}} f(x)dxdy\right| < \varepsilon\right) = \left(\left|\sum f(\xi_{v+b})\Delta x_{v+b}\Delta y_{v+b} - \iint_{S_{v+b}} f(x)dxdy\right| < \varepsilon\right).$$

That is to say:

$$\left(\left|\sum f(\xi_{u+a})\Delta x_{u+a}\Delta y_{u+a} - \iint_{S_{u+a}} f(x)dxdy\right| < \varepsilon\right) + \left(\left|\sum f(\xi_c)\Delta x_c\Delta y_c - \iint_{S_c} f(x)dxdy\right| < \varepsilon\right)$$
$$= \left(\left|\sum f(\xi_{v+b})\Delta x_{v+b}\Delta y_{v+b} - \iint_{S_{v+b}} f(x)dxdy\right| < \varepsilon\right) + \left(\left|\sum f(\xi_c)\Delta x_c\Delta y_c - \iint_{S_c} f(x)dxdy\right| < \varepsilon\right).$$

Hence:

$$\left(\left|\sum f(\xi_{u+a+c})\Delta x_{u+a+c}\Delta y_{u+a+c} - \iint_{S_{u+a+c}} f(x)dxdy\right| < \varepsilon\right) = \left(\left|\sum f(\xi_{v+b+c})\Delta x_{v+b+c}\Delta y_{v+b+c} - \iint_{S_{v+b+c}} f(x)dxdy\right| < \varepsilon\right).$$

(iv) Defining the naturally existing canonical map $\theta: \frac{S/I}{J} \to {}^{S/I}\!/_{J}\!/_{K}$

by:

$$\left(\left|\sum f(\xi_{u+a+c})\Delta x_{u+a+c}\Delta y_{u+a+c} - \iint_{S_{u+a+c}} f(x)dxdy\right| < \varepsilon\right) = \left(\left|\sum f(\xi_{u+a+c+d})\Delta x_{u+a+c+d}\Delta y_{u+a+c+d} - \iint_{S_{u+a+c+d}} f(x)dxdy\right| < \varepsilon\right).$$

Then, the Onto-Homomorphism can be shown as follows:

The preservation of composition is: $\theta\left[\left(\left|\sum f(\xi_{u+a+c})\Delta x_{u+a+c}\Delta y_{u+a+c} - \iint_{S_{u+a+c}} f(x)dxdy\right| < \varepsilon\right) + \left(\left|\sum f(\xi_{v+b+c})\Delta x_{v+b+c}\Delta y_{v+b+c} - \iint_{S_{v+b+c}} f(x)dxdy\right| < \varepsilon\right)\right]$ Then, by triangular inequality, $\theta\left(\left|\sum f\left(\xi_{(u+v)+(a+b)+c}\right)\Delta x_{(u+v)+(a+b)+c}\Delta y_{(u+v)+(a+b)+c} \iint_{S_{(u+v)+(a+b+c)}} f(x)dxdy\right| < \varepsilon\right)$.

Which by definition of θ is: $\left(\left|\sum f\left(\xi_{(u+v)+(a+b)+c+d}\right)\Delta x_{(u+v)+(a+b)+c+d}\Delta y_{(u+v)+(a+b)+c+d} - \iint_{S_{(u+v)+(a+b)+c+d}} f(x)dxdy\right| < \varepsilon\right)$

$$\Rightarrow \left(\left|\sum f(\xi_{u+a+c+d})\Delta x_{u+a+c+d}\Delta y_{u+a+c+d} - \iint\limits_{S_{u+a+c+d}} f(x)dxdy\right| < \varepsilon\right) + \left(\left|\sum f(\xi_{v+b+c+d})\Delta x_{v+b+c+d}\Delta y_{v+b+c+d} - \iint_{S_{v+b+c+d}} f(x)dxdy\right| < \varepsilon\right)$$

$$\Rightarrow \theta\left(\left|\sum f(\xi_{u+a+c})\Delta x_{u+a+c}\Delta y_{u+a+c} - \iint_{S_{u+a+c}} f(x)dxdy\right| < \varepsilon\right) = \theta\left(\left|\sum f(\xi_{v+b+c})\Delta x_{v+b+c}\Delta y_{v+b+c} - \iint_{S_{v+b+c}} f(x)dxdy\right| < \varepsilon\right),$$ as required.

To show Onto-ness.:

Let $\theta^{-1}\left(\left|\sum f(\xi_{u+a+c+d})\Delta x_{u+a+c+d}\Delta y_{u+a+c+d} - \iint_{S_{u+a+c+d}} f(x)dxdy\right| < \varepsilon\right) = \theta^{-1}\left(\left|\sum f(\xi_{v+b+c+d})\Delta x_{v+b+c+d}\Delta y_{v+b+c+d} - \iint_{S_{v+b+c+d}} f(x)dxdy\right| < \varepsilon\right)$.

Then, by definition, $\left(\left|\sum f(\xi_{u+a+c})\Delta x_{u+a+c}\Delta y_{u+a+c} - \iint_{S_{u+a+c}} f(x)dxdy\right| < \varepsilon\right) = \left(\left|\sum f(\xi_{v+b+c})\Delta x_{v+b+c}\Delta y_{v+b+c} - \iint_{S_{v+b+c}} f(x)dxdy\right| < \varepsilon\right)$. That $\left(\left|\sum f(\xi_{u+a+c})\Delta x_{u+a+c}\Delta y_{u+a+c} - \iint_{S_{u+a+c}} f(x)dxdy\right| < \varepsilon\right) + \left(\left|\sum f(\xi_d)\Delta x_d\Delta y_d - \iint_{S_d} f(x)dxdy\right| < \varepsilon\right) = \left(\left|\sum f(\xi_{v+b+c})\Delta x_{v+b+c}\Delta y_{v+b+c} - \iint_{S_{v+b+c}} f(x)dxdy\right| < \varepsilon\right) + \left(\left|\sum f(\xi_d)\Delta x_d\Delta y_d - \iint_{S_d} f(x)dxdy\right| < \varepsilon\right)$.

That is, $\left(\left|\sum f(\xi_{u+a+c+d})\Delta x_{u+a+c+d}\Delta y_{u+a+c+d} - \iint_{S_{u+a+c+d}} f(x)dxdy\right| < \varepsilon\right) = \left(\left|\sum f(\xi_{v+b+c+d})\Delta x_{v+b+c+d}\Delta y_{v+b+c+d} - \iint_{S_{v+b+c+d}} f(x)dxdy\right| < \varepsilon\right)$.

CONCLUSION

From the above results therefore, one can conclude that:

$$\left(\left|\sum f(\xi_0)\Delta x_0\Delta y_0 - \iint_{S_0} f(x)dxdy\right| < \varepsilon\right) \trianglelefteq \left(\left|\sum f(\xi_1)\Delta x_1\Delta y_1 - \iint\limits_{S_1} f(x)dxdy\right| < \varepsilon\right) \trianglelefteq \dots$$

$$\left(\left|\sum f(\xi_{abcd})\Delta x_{abcd}\Delta y_{abcd} - \iint\limits_{S_{abcd}} f(x)dxdy\right| < \varepsilon\right) \trianglelefteq$$
$$\left(\left|\sum f(\xi_{abc})\Delta x_{abc}\Delta y_{abc} - \iint\limits_{S_{abc}} f(x)dxdy\right| < \varepsilon\right) \trianglelefteq$$
$$\left(\left|\sum f(\xi_{ab})\Delta x_{ab}\Delta y_{ab} - \iint\limits_{S_{ab}} f(x)dxdy\right| < \varepsilon\right) \trianglelefteq$$
$$\left(\left|\sum f(\xi_{a})\Delta x_{a}\Delta y_{a} - \iint_{S_{a}} f(x)dxdy\right| < \varepsilon\right);$$

Forms the composition series of solvability of $\left|\sum f(\xi_{a})\Delta x_{a}\Delta y_{a} - \iint_{S_{a}} f(x)dxdy\right| < \varepsilon$ whenever $|\Delta x_a \Delta y_a| < \delta$; where $f(\xi_a) \in \Delta x_a \Delta y_a$ and ε and δ are infinitesimally small numbers.

REFERENCES

1. Newton, S. I. (1687). *Philosophiae Naturalis Principia Mathematica.* Journal of the London Mathematical Society, London.
2. Suschkewitz, A.K. (1928). Uber Die Endlichen Gruppen Ohne Das Gesetz Der
3. Eindeutigen Umkehrbarkeit. *Math.Ann*., **99:** 30-50.

a. Cauchy, A. L. (1932). Oeuvres Completes. *Gauthier-Villars*, Paris, **13**(2): 171 – 282.

4. Cayley, A. (1854). On the Theory of Groups, as Depending on the Symbolic Equations $\theta^n = 1$. *Philosophical Magazine,* 4th series, **7**(42): 40 – 47.
5. Galois, E. (1846). Theory of Equations, Journal de Mathematique Pures et Appliquees,Bourg-La-Reine. Retrieved in 2020 at http://www.mathshistory.st.andrews.ac.uk/Biographies/Galois.
6. Heinstein, I.N.(1964). Topics in Algebra. John Wiley and Sons, 2nd Edition.
7. Joseph, J**.** R**.** (2005). *A First Course in Abstract Algebra*. Prentice Hall, Upper Saddle River, New Jersey.
8. Dummit, D.S.and Foote,R.M. (1990). *Abstract Algebra*. Prentice-Hall, New York, USA.