NUMERICAL APPROXIMATION METHODS FOR SOLVING INTEGRO-DIFFERENTIAL EQUATIONS VIA THIRD KIND CHEBYSHEV AND LAGUERRE POLYNOMIALS

AYINDE, A. M.[1*], JIMOH, A. K.[2], MOMOH, A. A.[1] AND ALIYU. H. B.[1]

[1]Department of Mathematics, Modibbo Adama University, Yola, Nigeria.

[2]Department of Statistics and Mathematical Sciences, Faculty of Pure and Applied Sciences, Kwara State University , Malete, Nigeria.

[*]Corresponding author email: abdullahim@mautech.edu.ng

ABSTRACT

In this work, the collocation method via third kind Chebyshev and Laguerre polynomials as basis functions were developed and used to solve Volterra integro-differential equations (IDEs) using the standard collocation method. An assumed approximate solution is substituted into the given problem, thus resulted in more unknown constants to be determined. After simplification and collocations, resulted in linear algebraic equations which are then solved via maple 18 to obtain unknown constants that are involved. Comparisons were made with the two trial solutions mentioned above in terms of errors obtained. Numerical examples were given to illustrate the performance of the method for various orders. However, the third kind Chebyshev polynomial basis exhibits better accuracy over the Laguerre polynomials as can be seen from the tables of errors presented.

Keywords: Collocation Method, Volterra Intrgro-differential Equations, Chebyshev Polynomial, and Laguerre Polynomial.

INTRODUCTİON

Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. Consequently, there had been extraordinary enthusiasm by several authors towards obtaining the numerical solutions of this class of problems. In literature, there exist numerous numerical techniques to solve Integro-differential equation such as Wavelet-Galerkin Method (WGM) by (Avudainayan & Vani 2000), Homotopy Analysis Method (HAM) by (Kunjan Shah & Twinkle Singh, 2015).

Furthermore, the application of the Taylor, Chebyshev, Hermite, Legendre, and Laguerre polynomials and their numerical merits in solving integral and integro-differential equations (IDEs) numerically have been discussed in (Akyuz & Sezer, 2003), (Maleknejad & Mahmoudi, 2003), (Taiwo, O. A., Alim, A. T. & Akanmu, M. A., 2014), and (Richard & Roderick, 2010).

Legendre polynomials first arose in the problem of expressing the Newtonian potential of a conservative force field in an infinite series involving the distance variable of two points and their included Centre angle. Other similar problems dealing with either gravitational potential or electrostatic potential and steady-state heat conduction problems in spherical solids, also lead to Legendre polynomials. Other polynomials which commonly occur in applications are Laguerre and Hermite. They play an important role in quantum mechanics, and in probability theory but the focus of this research work is on third kind Chebyshev and Laguerre polynomials for solving Volterra Integro-Differential Equations (VIDEs).

Also, many techniques such as a new algorithm for calculating Adomian polynomials, (Hashim, 2006), Chebyshev polynomials by (Eslahchi, M. R., Mehdi, D. & Sanaz, A., 2012), Chebyshev and Legendre by (Abubakar & Taiwo, 2014), Homotopy Perturbation Method (HPM), (Wazwaz, 2011) and, Variation Iteration Decomposition Method (VIDM) by (Ignatius & Mamadu, 2016). Application of Adomian's decomposition method on Integro-differential equation also examined by (Wazwaz, 2001) and others have been used to derive solutions of some classes of integro-differential equations.
The great work did by the researchers aforementioned motivated us and eventually led to the proposal of a numerical approximation method that is efficient and accurate with less computational work to obtain an approximate solution of high order linear Volterra integro-differential equations of the form

$$P_{01}\varphi^{m}(z) + P_{11}\varphi^{m-1}(z) + \cdots + P_{m-1}\varphi^{'}(z) + P_{m}\varphi(z) + \lambda \int_{h(z)}^{i(z)} K(z,s)\varphi(t)dt = f(z) \quad (1)$$

Subject to the conditions

$$\varphi(0) = C_{0}, \varphi^{'}(0) = C_{1}, \varphi^{''}(0) = C_{2}, \cdots, \varphi^{(m-1)}(0) = C_{m-1} \quad (2)$$

where $P's$ are real constants; i, h are finite constants; $K(z,s)$, and $f(z)$ are given real-valued functions and $\varphi's$ are unknown constants to be determined.

The present work is aimed at producing exact and approximate solutions with less cumbersome and easy to handle by third kind Chebyshev and Laguerre polynomials methods, thus, the main objectives are to transform the integro-differential equation ine equation (1) subject to initial conditiobns in equation (2) into a system of linear algebraic equations, obtain the solution of the linear algebraic equation, test the efficiency of the methods on some numerical examples, and compare the two methods with each other.

BASIC DEFINITIONS

Definition 2.1:

Integro-Differential Equation - an Integro-Differential Equation (IDE) is an equation in which the unknown function $\varphi(z)$ appears under the integral sign and contains an ordinary derivative $\varphi^{(m)}$ as well. A standard integro-differential equation is of the form:

$$\varphi^{(m)}(z) = f(z) + \lambda \int_{h(z)}^{i(z)} K(z,s)\varphi(s)ds \quad (3)$$

Where $i(z)$ and $h(z)$ are limits of integration which may be constants, variables or combined. $\lambda$is a constant parameter, $f(z)$ is a given function and $K(z,s)$ is a known function of two variables z and s, called the kernel.

We have Fredholm integro-differential equation if the upper limit of integration is a constant and it is called Volterra integro-differential equation if the limit $\varphi(z)$ is replaced with a variable of integration z.

Definition 2.2:

Collocation Method: is a method involving the determination of an approximate solution in a suitable set of functions sometimes called trial solution and also is a method of evaluating a given differential equation at some points to nullify the values of an ordinary differential equation at those points.

Definition 2.3:

Exact Solution: a solution is called an exact solution if it can be expressed in a closed form, such as a polynomial, exponential function, trigonometric function, or the combination of two or more of these elementary functions.

Definition 2.4:

Approximate Solution: an approximate solution is an inexact representation of the exact solution that is still close enough to be used instead of exact and it is denoted by $\zeta_M(z)$, where M is the degree of the approximant used in the calculation. Methods of the approximate solution are usually adopted because complete information needed to arrive at the exact solution may not be given. In this work, the approximate solution used is given as

$$\zeta_m(z) = \sum_{i=0}^{M} c_i \varphi_i(z) \quad (4)$$

where$c_{i,}$ $i = 0, 1, 2, \ldots, M$ are unknown constants to be determined, $\varphi_i(z)(i \geq 0)$ is the basis functions which is either the third kind of Chebyshev or Laguerre Polynomials and M is the degree of approximating Polynomials.

Definition 2.5:

The third kind Chebyshev Polynomial in [-1, 1] of degree m is denoted by $V_m(z)$ and defined by

$$V_m(z) = \cos\frac{\left(m + \frac{1}{2}\right)\vartheta}{\cos\left(\frac{\vartheta}{2}\right)},$$

$$\text{where} z = \cos\vartheta \quad (5)$$

This class of Chebyshev Polynomials satisfies the following recurrence relation

$$V_0(z) = 1, \quad V_1(z) = 2z - 1, V_m(z) = 2zV_{m-1}(z) - V_{m-2}(z), \; m = 2, 3, \cdots \quad (6)$$

The third kind Chebyshev Polynomial in $[\alpha, \beta]$ of degree m, is denoted by $V_m^*(z)$ and is defined

by $V_m^*(z) = \cos\frac{\left(m+\frac{1}{2}\right)\vartheta}{\cos\left(\frac{\vartheta}{2}\right)}$,

$$\cos\vartheta = \frac{2z - (\alpha + \beta)}{\beta - \alpha} \vartheta \epsilon [0, \pi] \quad (7)$$

All the results of Chebyshev polynomials of the third kind can be easily transformed to give the corresponding results for their shifted ones. The orthogonality relations of $V_m^*(z)$ on $[\alpha, \beta]$ with respect to the weight functions $\sqrt{\frac{z-\alpha}{\beta-z}}$ is given by

$$\int_{\alpha}^{\beta} \sqrt{\frac{z-\alpha}{\beta-z}} = \begin{cases} (\beta - \alpha)\frac{\pi}{2}, m = n \\ 0, \; m \neq n \end{cases}$$ Doh, et al. (2015)

Definition 2.6:

The Laguerre polynomials are defined as

$$L_m(z) = \sum_{j=0}^{M} (-1)^j \frac{m!}{(j!)^2 (m-j)!}$$

$$= \sum_{j=0}^{M} (-1)\frac{1}{j!}\binom{m}{j} z^j,$$

$$m = 0, 1, 2, \cdots. \quad (8)$$

The recurrence relation is

$$L_m(z) = \frac{e^{\,z}}{m!}\frac{d^m}{dz^m}(e^{-z} z^m),$$

$$m = 2, 3, 4, \cdots. \quad (9)$$

where

$L_0(z) = 1, L_1(z) = 1 - x$.

Using this formula, the first few Laguerre Polynomials can be obtained as

$$L_2(z) = \frac{e^{z}}{2!}\frac{d^2}{dz^2}(e^{-z}z^2)$$

$$= \frac{1}{2}(z^2 - 4z + 2) \quad (10)$$

$$L_3(z) = \frac{e^{z}}{3!}\frac{d^3}{dz^3}(e^{-z}z^3)$$

$$= \frac{1}{3!}(6 - 18z - 9z^2 - z^3) \quad (11)$$

$$\vdots$$

$$m = \frac{1}{m!}\big((-z))^m + m^2(-z)^{m-1} + \cdots + m(m!)(-z) + m!\big)$$

and so on. Laguerre polynomials (2001)

Problem considered and methodology

In this section, we applied the standard collocation method to solve equation (1) using the following basis functions:

(i) Third kind Chebyshev Polynomials and

(ii) Laguerre Polynomials.

Standard Collocation Method by Third kind Chebyshev Polynomials

To solve the general problem given in equation (1) subject to the conditions given in equation (2) using the standard collocation method with third kind Chebyshev polynomials as basis functions, we assumed an approximate solution of the form:

$$\varphi(z) = \sum_{i=0}^{M} c_i V_i^*(z) \quad (12)$$

where $c_{i,}$ $i = 0, 1, 2, \ldots, M$ are unknown constants and $V_i^*(z)(i \geq 0)$are Chebyshev polynomials of the third kind defined in equation (5) to (7), M is the degree of approximating Polynomials, where in most cases the better approximate solution (i.e. closer to the exact solution) is produced by larger M, and $c_i$is the specialized coordinate called Degree of freedom.

Thus, differentiating equation (12) with respect to z m times, we obtain

$$\varphi'_m(z) = \sum_{i=0}^{M} c_i V_i^{*'}(z)$$

$$\varphi''_m(z) = \sum_{i=0}^{M} c_i V_i^{*''}(z)$$

$$\vdots$$

$$\varphi^{(m)}(z) = \sum_{i=0}^{M} c_i V_i^{*(m)}(z) \quad (13)$$

Hence, substituting equations (12) and (13) into equation (1), we obtain

$$P_{01}\sum_{i=0}^{M} c_i V_i^{(m)}(z) + P_{11}\sum_{i=0}^{M} c_i V_i^{(m-1)}(z) + P_{21}\sum_{i=0}^{M} c_i V_i^{(m-1)}(z) + \cdots + P_{m1}\sum_{i=0}^{M} c_i V_i^{(m)}(z) + \lambda\int_{h(z)}^{i(z)} K(z,s)\left(\sum_{i=0}^{M} c_i V_i(s)\right)ds = f(z) \quad (14)$$

Evaluating the integral part of the equation (14) to obtain

$$P_{01}\sum_{i=0}^{M} c_i V_i^{(m)}(z) + P_{11}\sum_{i=0}^{M} c_i V_i^{(m-1)}(z) + P_{21}\sum_{i=0}^{M} c_i V_i^{(m-1)}(z) + \cdots + P_{m1}\sum_{i=0}^{M} c_i V_i^{(m)}(z) + \lambda G(z) = f(z) \quad (15)$$

where $G(z) = \int_{h(z)}^{i(z)} K(z,s)\big(\sum_{i=0}^{M} c_i V_i(s)\big)ds$

Thus, collocating equation (15) at the point $z = z_j$, we obtain

$$P_{01}\sum_{i=0}^{M} c_i V_i^{(m)}(z_j) + P_{11}\sum_{i=0}^{M} c_i V_i^{(m-1)}(z_j)$$
$$+P_{21}\sum_{i=0}^{M} c_i V_i^{(m-1)}(z_j) + \cdots + P_{m1}\sum_{i=0}^{M} c_i V_i^{(m)}(z_j)$$
$$+\lambda G(z_j) = f(z_j) \qquad (16)$$

and

$$z_j = \alpha + \frac{(\beta-\alpha)j}{M+1}; j = 1,2,\dots,M \qquad (17)$$

Thus, equation (16) is then put into matrix form as

$$D\underline{z} = \underline{b} \qquad (18)$$

where

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} & & d_{1,m} \\ d_{21} & d_{22} & d_{23} & \cdots & d_{2,m} \\ d_{31} & d_{32} & d_{33} & & d_{3,m} \\ & \vdots & & \ddots & \vdots \\ d_{m,1} & d_{m,2} & d_{m,3} & \cdots & d_{m,m} \end{pmatrix} \qquad (19)$$

$$\underline{z} = (z_1, z_2, z_3, \cdots, z_m)^T \qquad (20)$$

$$\underline{b} = (f(b_1), f(b_2), f(b_3) \dots, f(b_m))^T \qquad (21)$$

Thus, equation (16) gives rise to $(M-1)$system of linear algebraic equations in $(M+1)$ unknown constants and m extra equations are obtained using the conditions given in equation (2). Altogether, we now have$(M+1)$ system of linear algebraic equations. These equations are then put in matrix form and solved via Maple 18 software to obtain $(M+1)$ unknown constants $c_i(i \geq 0)$which are then substituted back into the approximate solution given by equation (12).

Standard Collocation Method by Laguerre Polynomials

To solve the general problem given in equation (1) subject to the conditions given in equation (2) using the standard collocation method with Laguerre polynomials as basis functions, we assumed an approximate solution of the form:

$$Q_m(z) = \sum_{i=0}^{M} c_i L_i(z) \qquad (22)$$

where $c_{i,}$ $i = 0,1,2,\dots,M$ are unknown constants and $L_i(z)(i \geq 0)$ are Laguerre polynomials defined in equations (8) to (11), M is the degree of approximating Polynomials, where in most cases the better approximate solution (i.e. closer to the exact solution) is produced by larger M. Thus, differentiating equation (17) with respect to z m times, we obtain

$$Q'_m(z) = \sum_{i=0}^{M} c_i L'_i(z)$$
$$Q''_m(z) = \sum_{i=0}^{M} c_i L''_i(z)$$
$$\vdots$$
$$Q^{(m)}(z) = \sum_{i=0}^{M} c_i L_i^{(m)}(z) \qquad (23)$$

Hence, substituting equations (22) and (23) into equation (1), we obtain

$$Q_{01}\sum_{i=0}^{M} c_i L_i^{(m)}(z) + Q_{11}\sum_{i=0}^{M} c_i L_i^{(m-1)}(z)$$
$$+Q_{21}\sum_{i=0}^{M} c_i L_i^{(m-2)}(z) + \cdots + Q_{m1}\sum_{i=0}^{M} c_i L_i(z)$$
$$+\lambda\int_{h(z)}^{i(z)} K(z,s)\left(\sum_{i=0}^{M} c_i L_i(s)\right) ds = f(z) \qquad (24)$$

Evaluating the integral part of equation (24) to obtain

$$Q_{01}\sum_{i=0}^{M} c_i L_i^{(m)}(z) + Q_{11}\sum_{i=0}^{M} c_i L_i^{(m-1)}(z) + Q_{21}\sum_{i=0}^{M} c_i L_i^{(m-2)}(z) + \cdots + Q\sum_{i=0}^{M} c_i L_i(z) + \lambda G(z) = f(z), \quad (25)$$

where $G(z) = \int_{h(z)}^{i(z)} K(z,s)\left(\sum_{i=0}^{M} c_i L_i(s)\right)ds$

Thus, collocating equation (25) at the point $= z_j$, we obtain

$$Q_{01}\sum_{i=0}^{M} c_i L_i^{(m)}(z_j) + Q_{11}\sum_{i=0}^{M} c_i L_i^{(m-1)}(z_j) + Q_{21}\sum_{i=0}^{M} c_i L_i^{(m-2)}(z_j) + \cdots + Q\sum_{i=0}^{M} c_i L_i(z_j) + \lambda G(z_j) = f(z_j), \quad (26)$$

and

$$z_j = \alpha + \frac{(\beta-\alpha)j}{M+1};\ j = 1,2,\ldots,M \quad (27)$$

Thus, equation (15) is then put into matrix form as

$$P\underline{z} = \underline{d} \quad (28)$$

where

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & & p_{1,m} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2,m} \\ p_{31} & p_{32} & p_{33} & & p_{3,m} \\ & \vdots & & \ddots & \vdots \\ p_{m,1} & p_{m,2} & p_{m,3} & \cdots & p_{m,m} \end{pmatrix} \quad (29)$$

$$\underline{z} = (z_1, z_2, z_3, \cdots, z_m)^T \quad (30)$$

$$\underline{d} = (f(d_1), f(d_2), f(d_3) \ldots, f(d_m))^T \quad (31)$$

Thus, equation (26) gives rise to $(M-1)$system of linear algebraic equations in $(M+1)$ unknown constants and m extra equations are obtained using the conditions given in equation (2). Altogether, we have $(M+1)$ system of linear algebraic equations. These equations are put into matrix form and solved via Maple 18 software to obtain $(M+1)$ unknown constants $c_i(i \geq 0)$which are then substituted back into the approximate solution given by equation (22).

Numerical Examples and Results

In this section, we have demonstrated the standard collocation approximation method on high order integro-differential equations using Chebyshev polynomials of the third kind and Laguerre Polynomials as the basis functions. The results obtained are compared with each other on three problems to test for the effectiveness and efficiency of our methods via the Maple 18 software.

Numerical Example 1

Consider the second-order linear Volterra integro-differential equation

$$\varphi''(z) = 1 + \int_0^z (z-s)\varphi(s)ds. \quad (31)$$

with initial conditions

$$\varphi(0) = 1, \quad \varphi'(0) = 0 \quad (32)$$

The exact solution is given as

$$\varphi(z) = \cos h(z) \quad (33)$$

Wazwaz (2011).

Numerical Example 2

Consider the third-order linear Volterra integro-differential equation.

$$\varphi'''(z) = -1 + z - \int_0^z (z-s)\varphi(s)ds. \qquad (33)$$

with initial conditions

$$\varphi(0) = 1, \quad \varphi'(0) = -1, \quad \varphi''(0) = 1 \; (35)$$

The exact solution is given as

$$\varphi(z) = e^{-z} \qquad (36)$$

Wazwaz (2011).

Numerical Example 3

Consider the fourth-order linear Volterra integro-differential equation.

$$\varphi^{(iv)}(z) = 3e^z + e^{2z} - \int_0^z e^{2(z-s)}\varphi(s)ds. \quad (37)$$

with initial conditions

$$\varphi(0) = 0, \quad \varphi'(0) = 1, \quad \varphi''(0) = 2, \quad \varphi'''(0) = 3 \qquad (38)$$

The exact solution is given as

$$\varphi(z) = ze^z \qquad (39)$$

Wazwaz (2011).

Remark: We defined error as:

$$Error = |\varphi(z) - \varphi_M(z)|;$$

where, $\varphi(z)$is the exact solution and $\varphi_M(z)$ is our approximate solution obtained for the various values of M.

Table 1: Results and Errors obtained for Example 1:

Z	**Exact Result**	**Results by Chebyshev Polys. For Case M = 5**	**Results by Laguerre Polys. For Case M = 5**	**Error by Chebyshev Polys.**	**Error by Laguerre Polys.**
0.0	1.000000000	0.999999999	0.999999991	1.0000e-09	9.0000e-09
0.1	1.005004168	1.005001453	1.005001453	2.7150e-06	2.7150e-06
0.2	1.020066756	1.020059361	1.020059365	7.3910e-06	7.3910e-06
0.3	1.045338514	1.045326566	1.045326566	1.1948e-05	1.1948e-05
0.4	1.081072372	1.081055998	1.081055998	1.6374e-05	1.6374e-05
0.5	1.127625965	1.127605022	1.127605021	2.0943e-05	2.0944e-05
0.6	1.185465218	1.185439717	1.185439716	2.5501e-05	2.5502e-05
0.7	1.255169006	1.255139183	1.255139181	2.9823e-05	2.9825e-05
0.8	1.337434946	1.337399845	1.337399843	3.5101e-05	3.5103e-05
0.9	1.433086385	1.433039756	1.433039754	4.6629e-05	4.6631e-05
1.0	1.543080635	1.543002899	1.543002895	7.7736e-05	7.7740e-05

Table 2: Results and Errors obtained for Example 2:

Z	**Exact Result**	**Results by Chebyshev Polys. For Case M = 6**	**Results by Laguerre Polys. For Case M = 6**	**Error by Chebyshev Polys.**	**Error by Laguerre Polys.**
0.0	1.000000000	1.000000000	0.999999991	0.00000000	5.0000e-10
0.1	0.904837418	0.904837455	0.904837453	3.6500e-08	3.4800e-08
0.2	0.8187307531	0.818730963	0.818730960	2.0980e-07	2.0690e-07
0.3	0.7408182207	0.740818754	0.740818741	5.3310e-07	5.2900e-07
0.4	0.6703200460	0.670321047	0.670321042	1.0015e-06	9.9620e-07
0.5	0.6065306597	0.606532278	0.606532272	1.6185e-06	1.6119e-06
0.6	0.5488116361	0.548814017	0.548814009	2.3809e-06	2.3732e-06
0.7	0.4965853038	0.496588597	0.496588588	3.2932e-06	3.2843e-06
0.8	0.4493289641	0.449333439	0.449333429	4.4754e-06	4.4652e-06
0.9	0.4065696597	0.406576081	0.406569651	6.4217e-06	6.4103e-06
1.0	0.3678794412	0.367889905	0.367889893	1.0464e-05	1.0451e-05

Table 3: Results and Errors obtained for Example 3:

z	**Exact Result**	**Results by Chebyshev Polys. For Case M = 6**	**Results by Laguerre Polys. For Case M = 6**	**Error by Chebyshev Polys.**	**Error by Laguerre Polys.**
0.0	0.000000000	7.2070e-10	5.6300e-07	7.2070e-10	5.6300e-07
0.1	0.110517092	0.110517177	0.110516751	8.5200e-08	3.3190e-07
0.2	0.244280552	0.244281613	0.244281300	1.0609e-06	7.4850e-07
0.3	0.404957642	0.404961911	0.404961669	4.2686e-06	4.0264e-06
0.4	0.596729879	0.596740885	0.596740686	1.1006e-05	1.0807e-05
0.5	0.824360635	0.824383233	0.824383054	2.2597e-05	2.2419e-05
0.6	1.093271280	1.093311589	1.093311417	4.0309e-05	4.0137e-05
0.7	1.409626895	1.409690973	1.409690797	6.4308e-05	6.3902e-05
0.8	1.780432743	1.780521604	1.780521419	8.8861e-05	8.8676e-05
0.9	2.213642800	2.213740116	2.213739911	9.7316e-05	9.7120e-05
1.0	2.718281828	2.718329139	2.718328935	4.7311e-05	4.7107e-05

Table 1, 2, and, 3 show the numerical solution obtained in terms of approximate solution and the errors for the linear integro-differential equations solved through third kind Chebyshev and Laguerre Polynomials basis function. We also observed from the examples solved that both methods converge close to the exact solution in a view iterations and lower error.

CONCLUSION

In this work, we have demonstrated the collocation approximation method for solving high-order Volterra integro-differential equations through the third kind Chebyshev Polynomials and Laguerre Polynomials as basis functions and compared the results obtained with each other. The results obtained by the third kind Chebyshev Polynomials as basis functions get closer to the exact solution than the results by Laguerre Polynomials in some examples.

However, we also observed that the results obtained yield a good approximation to the exact solution only in a few iterations in all the problems considered (as can be seen from tables of results). Thus, we conclude that the methods are reliable and effective for the class of problems considered. This work is limited to linear integro-differential equation, it is therefore recommended for the immediate solution of other types of equations, for example, Fractional differential equations, Integro-differential difference equations, and Partial differential equations.

Acknowledgements

The authors appreciate the reviewers and editor for their careful reading, valuable suggestions and timely review.

References

Abubakar, A. &Taiwo, O. A. (2014). Integral collocation approximation methods for the numerical solution of high-orders linear Fredholm-Volterra integro-differentialequations. *Americal Journal of Computational and Applied Mathematics*, 4(4), 111-117.

Akyuz, A. &Sezer, M. (2003). Chebyshev polynomial solutions of system of higher order linear differential equations with variable coefficients. *Applied Mathematics and Computer*, 144, 237-247.

Avudainayagam, A. &Vani, C. (2000). Wavelet-Galerkin method for integro-differential equations. *Applied Numerical Mathematics*, 32, 247-254.

Eslahchi, M. R., Mehdi, D. &Sanaz, A. (2012). The third and fourth kinds of Chebyshev polynomials and best uniform approximation. *Journal of Mathematical and Computer Modeling*, 55, 1746-1762.

Hashim, I. (2006). Adomian decomposition method for solving BVPs for fourth-order integro-differential equations. *Applied Mathematics and Computer*, 193, 658-664.

Ignatius, N. N. &Mamadu, E. (2016). Variational iteration decomposition method for numerical solution of integro-differential equations. *Nigerian Journal of Mathematics*, 25, 122-130.

Kunjan, S. & Twinkle, S. (2015). Solution of second kind Volterra integral and integro-differential equation by Homotopy Analysis Method. *International Journal of Mathematics*, 6(4), 49-59.

Maleknejad, F.&Mahmoudi, Y. (2003). Taylor polynomial solution of higher-order nonlinear Volterra – Fredholm integro-differential equations. *Applied Mathematics and Computer*, 155, 317-328.

Richard, B. & Roderick W. (2010). Special Functions. A graduate text, United States of America. *Cambridge University Press, New York*.

Taiwo, O. A., Alim, A. T. &Akanmu, M. A. (2014). Numerical Solutions for linearFredholmintegro-differential difference equations with variable coefficientsby collocation methods. *Journal of Educational Policy and Entrepreneurial Research*, 2(1), 175-185.

Wazwaz, A. M. (2001). A reliable algorithm for solving boundary value problems for higher-order integro-differential equations, *Applied Mathematics and Computer*, 118, 327-342.

Wazwaz, A. M. (2011). *Linear and Nonlinear Integral Equations*, (c).Higher Education Press, Beijing and Springer- Verlag Berlin.