A NOTE ON FULL TRANSFORMATION SEMIGROUP

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ABSTRACT

The topics in this paper deal with generating subsemigroups of the full transformation semigroup $T(X)$ on a totally ordered finite set $(X, \leq)$ with an arbitrary equivalence relation E defined on it. Using the equivalence relation on the set and the cardinality of the set, we investigate under what conditions some of these subsemigroups of $T(X)$ generated are equal.

Key words: Transformation semigroup, Equivalence relation, Totally ordered set, Equal

INTRODUCTION

Let S be a semigroup, then a nonempty subset T of S is a subsemigroup of S if $T^2 \subseteq T$.
Let X be a nonempty countable and finite set which is totally ordered and let $T(X)$ denote the full transformation semigroup, that is, the semigroup of all mappings $\alpha: X \to X$ under the usual composition. By composition of functions is it well-known that $T(X)$ is a regular semigroup (Howie, 1966). Moreover for some nonempty set X every semigroup can be embedded in $T(X)$ (Howie, 1966).

As usual, we consider α to be order-decreasing (order-increasing) if $\alpha(x) \leq x$ $(\alpha(x) \geq x)$ for all x in $Dom\alpha$ (for all x in X) ,and α is order-preserving if $x \leq y$ implies $\alpha(x) \leq \alpha(y)$ for all x, y in $Dom\alpha$ (for all x, y in X). The semigroup of all order-decreasing (order-increasing) full transformations is denoted by $D(X)$ $(I(X))$, while the semigroup of all order-preserving full transformations is denoted by $O(X)$.

The full transformation semigroup is the semigroup analogue of the symmetric group defined on a nonempty set (Howie, 1966).
The notion of (full) transformation semigroup has been extensively researched. You in (You, 2002) determined all the maximal regular subsemigroups of all ideals of the fnite full transformation semigroup. In (Yang and Yang, 2004), Yang and Yang completely described the maximal subsemigroups of ideals of the fnite full transformation semigroup. In (Zhao et al, 2014), the authors showed that any maximal regular subsemigroup of ideals of the fnite full transformation semigroup is idempotent generated. But according to East *et al.*, (2015), the authors classified the maximal subsemigroups of $T(X)$when X is a infinite

set containing certain subgroups of the symmetric group on X.

Many subsemigroups of the (full) transformation semigroup $T(X)$, in several direction, have been investigated by many researcher, like (Garba *et al*., 2017; Sun, 2013; Sun and Han, 2016; Dimitrova and Koppitz, 2011; Kemal and Hayrullah, 2018). In (Araujo and Konieczny, 2013), the authors provided a description of $C(\alpha)$ for a general$\alpha \in T(X)$, where X is an arbitrary set (finite or infinite) and $C(\alpha)$ the centralizer of α. In (Adeniji and Makanjuola, 2013), the author considered the certain full transformation Semigroup for congruence.

The combinatorial nature notion of subsemigroups of the (full) transformation semigroup has been studied in various directions (Howie, 1971; Laradji and Umar, 2004; Higgins, 1995; Umar, 1997; Gomes and Howie, 1992; Garba,1994; Umar, 1992, Zubairu and Bashir, 2018). Also in Pie and Zhon, (2011) studied subsemigroup of the (full) transformation semigroup and proved that the subsemigroup $T_E(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Longrightarrow (\alpha(x), \alpha(y)) \in E\}$ is an abundant semigroup but not a regular semigroup if the equivalence relation, E is simple.

For important text in semigroup the following are good references (Ganyushkin and Mazorchuk, 2009).

METHODOLOGY

This research is motivated by the fact that if the equivalence relation E on nonempty totally ordered set X is the universal equivalence relation then $T(X) = T_E(X)$ and if E is the diagonal equivalence relation then $S(X) = T_E(X)$, where $S(X)$ consist of all injective maps from X into X , studied by Konieczny in (Konieczny, 2010: 2011).

Let E be an arbitary equivalence relation defined on the totally ordered set $(X, \leq)$. The aim of this work is to study the intersections of these subsemigroups $T_E(X)$, $D(X)$, $OR(X)$ and $OE(X)$ of $T(X)$:

- $D(X) = \{\alpha \in T(X) : \forall x \in X, \alpha(x) \leq x\}$
- $T_E(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Longrightarrow (\alpha(x), \alpha(y)) \in E\}$
- $OR(X) = \{\alpha \in T(X) : \forall x, y \in X, x \leq y, \Longrightarrow \alpha(x) \geq \alpha(y)\}$
- $OE(X) = \{\alpha \in T(X) : \forall x, y \in X, x \leq y, \Longrightarrow (\alpha(x), \alpha(y)) \in E\}$,

thereby investigating under what conditions some of their intersections are equal.

All the aforementioned subsemigroups clearly contain the identity map on X (i.e., are moniods), except $OE(X)$ and $OR(X)$ which generally do not. In proposition 2 it is shown under what condition $OE(X)$ is a moniod. But trivially $OR(X)$ is a monoid if and only if X is a singleton set.

We denote the equivalence class of $x \in X$ by $[x]$ and the set of all equivalence class on the set X by X/E for an arbitrary equivalence relation E on X. Also, $|X|$ will denote the cardinality of X.

RESULT AND DISCUSSION

Generally, it is known that the intersection of algebraic structures is either an algebraic structure or the empty set. Therefore, in this section we investigate under what conditions the intersections of the above named subsemigroups are equal to themselves.

Proposition 1

Let $T_E(X)$ and $D(X)$ be as defined. Then, the subsemigroup, $DE(X) = \{\alpha : \alpha(x) \leq x \Longrightarrow (\alpha(x), x) \in E\}$ of $T(X)$ is the intersection of $T(X)$ and $D(X)$. i.e. $DE(X) = T_E(X) \cap D(X)$.

Proof

Given that $\alpha \in DE(X)$, clearly, $\alpha \in D(X)$ for all $x \in X$. Now we show that $\alpha \in T_E(X)$. Assume that $(x, y) \in E$ for $x, y \in X$ such that $\alpha(x) \leq x$ and $\alpha(y) \leq y$ $(x, y) \in E$ then $(\alpha(x), \alpha(y)) \in E$. Thus $\alpha \in T_E(X)$. Hence $DE(X) \subseteq D(X) \cap T_E(X)$.

For the converse inclusion, let $\alpha \in D(X) \cap T_E(X)$. It follows that for all $, y \in X$, $\alpha(x) \leq x$, $\alpha(y) \leq y$, and $(x, y) \in E$ implies $(\alpha(x), \alpha(y)) \in E$. Thus, to prove that $\alpha \in DE(X)$, we need only show that for all $x \in X, (\alpha(x), x) \in E$. Clearly, if $\alpha(x) = x$, then $\alpha(x) \leq x \Longrightarrow (\alpha(x), x)$. On the other hand for $t \leq x$ if $\alpha(x) = t \neq x$ then since E is an equivalence relation and $\alpha \in T_E(X)$ then $\alpha(x) \leq x \Longrightarrow (\alpha(x), x) \in E$. So, $\alpha \in DE(X)$. Therefore $DE(X) \supseteq T_E(X) \cap D(X)$. Hence, $DE(X) = T_E(X) \cap D(X)$.

Corollary 1

For the subsemigroup $E(X) = \{\alpha \in T(X) : (x, y) \in E \Longrightarrow (\alpha(x), x) \in E \wedge (\alpha(y), y) \in E\}$ of $T(X)$, $E(X) \cap D(X) = T_E(X) \cap D(X)$.

Proof:

The proof follows easily from Proposition 1 above.

Corollary 2

$D(X) = DE(X)$ if and only if E is the universal equivalence relation.

Proof:

Suppose E is the universal equivalence relation on X, it follows easily that $DE(X) \subseteq D(X)$ since $DE(X) = T_E(X) \cap D(X)$.

For the reverse inclusion, assume $\alpha \in D(X)$, then for all $x \in X, \alpha(x) \leq x$. Clearly for all $x, y \in X$, $[x] = [y]$, it follows that $(\alpha(x), x) \in E$. Therefore $\alpha(x) \leq x \Longrightarrow (\alpha(x), x) \in E$, and so $\alpha \in DE(X)$. Hence $DE(X) \supseteq D(X)$

Conversely, assume to the contrary that E is not the universal equivalence relation, then there exist $A, B \in X/E$ such that $A \neq B$. Let $x \in A$ and $y \in B$ such that $x < y$, and $\alpha \in T(X)$ defined by

$$\alpha(b) = \begin{cases} a & ; \quad b > a, \\ b & ; \quad otherwise \end{cases}$$

then it follows clearly that $\alpha \in D(X)$, since $\alpha(x) \leq x$ for all $x \in X$ but $\alpha \notin$

$DE(X)$ since if $\alpha(x) = x$ and $\alpha(y) = x$, $(\alpha(y), y) \notin E$. This is a contradiction. Hence $D(X) \neq DE(X)$.

Proposition 2

The subsemigroup, $OE(X)$ is monoid if and only if the equivalence relation E defined on X is the universal equivalence relation.

Proof:

Suppose that $E \neq X \times X$. Let x and y be any two elements in X, such that $x > y$. Since E is not the universal equivalence relation, we assume that $(x, y) \notin E$. Therefore, by the definition of $OE(X)$, we must have that if $\alpha(x) = x$, then $\alpha(y) \neq y$ since $(x, y) \notin E$. Hence $\alpha \in OE(X)$ must be different from the identity transformation. Therefore since α is chosen arbitrarily, then $OE(X)$ is not a monoid.

Conversely, assuming that $E = X \times X$, then for any two elements x and y in X, $(x, y) \in E$. It easily follows that $\alpha(x) = x$ and $\alpha(y) = y$ is an element of $OE(X)$. Since $x, y \in X$, were chosen arbitrarily then α is the identity transformation and hence $OE(X)$ is a monoid.

Proposition 3

Let X be a nonempty set. Then $T_E(X) \cap OE(X) \neq \emptyset$ if and only if E is the universal equivalence relation, then

Proof.

Suppose $E \neq X \times X$, and assume $\alpha \in T_E(X) \cap OE(X)$. Therefore, for all $x < y \in X$, and $\alpha \in T_E(X)$ it follows that $(\alpha(x), \alpha(y)) \in E$. But since $\alpha \in OE(X)$ then for all $x < y \in X$, $(x, y) \in E$, a contradiction. Hence $T_E(X) \cap OE(X) \neq \emptyset$.

Conversely, if $T_E(X) \cap OE(X) = \emptyset$ then E is not reflexive and hence E is not the universal equivalence relation.

Corollary 3

If E is the universal equivalence relation, then $T_E(X) = OE(X)$.

Proposition 4

Let X be a non empty set and E be the universal equivalence relation, then $D(X) \cap OE(X) = OE(X)$ if and only if $|X| = 1$.

Proof.

Assuming $|X| \neq 1$, let $x < y \in X$ such that $(x, y) \in E$. We define $\alpha \in T(X)$ by

$$\alpha(a) = \begin{cases} y & ; \quad a = x, \\ a & ; \ otherwise \end{cases}$$

Clearly, $\alpha \in OE(X)$. But since $\alpha(x) = y > x$, then it follows that $\alpha \notin D(X)$, thus $\alpha \notin D(X) \cap OE(X)$.

The converse follows easily.

Corollary 4

Let X be a non empty set and E be the universal equivalence relation, then $OE(X) \supseteq D(X)$ if and only if $|X| = 1$.

Remark 1

If E is the diagonal equivalence relation or the equivalence class of the least element in X is a singleton class, then the cardinality of the subsemigroup $D(X) \cap OE(X)$ is equal to 1.

For example for $X = \{x, y, z\}$ such that $x < y < z$, then $D(X) \cap OE(X) = \{\begin{pmatrix} x & y & z \\ x & x & x \end{pmatrix}\}$.

Proposition 5

Let X be a nonempty set. Then $D(X) \cap OE(X) = D(X)$ if and only if E is the universal equivalence relation.

Proof.

suppose $E = X \times X$, then by corollary 3 we have that $D(X) \cap OE(X) = D(X) \cap T_E(X) = D(X) \cap T(X) = D(X)$. Hence, if $E = X \times X$, then $D(X) \cap OE(X) = D(X)$.

Conversely, assume $E \neq X \times X$, then there exists $x, z \in X$ such that $(x, z) \notin E$. Suppose $x \leq z$ and define $\alpha \in T(X)$ by

$$\alpha(a) = \begin{cases} x & ; \quad (x, a) \in E \\ z & ; \quad otherwise \end{cases}$$

Let $y \in X$ and $y \geq x$ such that $(x, y) \in E$. Then, it follows easily that $\alpha \in D(X)$. Therefore, for $y \leq z$, since $(\alpha(y), \alpha(z)) = (x, z) \notin E, \alpha \notin OE(X)$. Hence $D(X) \cap OE(X) \neq D(X)$.

We note that it easily follows that if $E \neq X \times X$, then $I_X \notin OE(X)$, where I_X is the identity map. Since $I_X \in D(X)$, we deduce that $D(X) \cap OE(X) \neq D(X)$.

Proposition 6

Let X be a nonempty set. Then $OR(X) \cap OE(X) = OR(X)$ if and only if E is the universal equivalence relation.

Proof.

Assuming $E \neq X \times X$. Let $A, B \in X/E$, with $A \neq B$. For $a, b \in X$ such that $a \in A$ and $b \in B$, we define $\alpha \in T(X)$ by

$$\alpha(y) = \begin{cases} \min(a, b) & ; \quad a < y, \\ \max(a, b) & ; \quad otherwise \end{cases}$$

Therefore, given that $x \leq y$, for $x, y \in X$, it clearly follows that for $a < x \leq y$ or $x \leq y \leq a$, $\alpha \in OR(X)$. Furthermore, in the case $x \leq a < y$, $\alpha(x) = max(a, b) > min(a, b) = \alpha(y)$, thus $\alpha \in OR(X)$. Suppose $d \in X$ such that $a < d$, then $(\alpha(a), \alpha(d)) = (max(a, b), min(a, b)) \notin E$. Hence, $\alpha \notin OE(X)$.

Conversely, suppose $E = X \times X$, then by corollary 3 we have that $OR(X) \cap OE(X) = OR(X) \cap TE(X) = OR(X) \cap T(X) = OR(X)$.

Hence, if $E = X \times X$, then $OR(X) \cap OE(X) = OR(X)$.

Proposition 7

Let X be a nonempty set and E is the diagonal equivalence relation, then $OR(X) \cap OE(X) = OE(X)$.

Proof

Assume that $E = 1_{X\times X}$, then we have that $OE = \{\alpha \in T(X): \alpha(x) = \alpha(y), \forall x, y \in X\}$. Therefore, we show that $\alpha \in OR(X)$. Let $x, y \in X$, such that $x \leq y$, and $\alpha(x) = \alpha(y)$ then it follows that $\alpha(x) \geq \alpha(y)$. Therefore, we deduce that $\alpha \in OR(X)$. Hence we have that $OR(X) \cap OE(X) = OE(X)$ whenever E is the diagonal equivalence relation.

CONCLUSION

In this work we studied and characterized the conditions under which the intersection some of the subsemigroups of the full transformation semigroup as presented are equal. The study considered that if an arbitrary equivalence relation, E defined on a nonempty totally ordered set of the full transformation semigroup is the universal or diagonal equivalence relation then the intersection of some subsemigroups of the full transformation semigroup equals their intersection.

Corollary 5

Let X be a nonempty set and E is the diagonal equivalence relation, then $OE(X) \subset OR(X)$.

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