EFFICIENCY OF URINE PUMPING IN THE HUMAN URETER

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Abstract

The efficiency of peristaltic pumping mechanism in the flow of urine from the kidneys to the bladder through the ureter is studied based on Liron's model, but with no applied external pressure gradient, in strict compliance with natural peristalsis situation as appears in physical organs. By applying the zeroth-order solutions, it is observed that in this study, the major contributing factor is the wave-plows (r) of the peristaltic motion of the flow tract rather than the Reynolds's number that has a direct bearing on the fluid property. But as would be expected, the Reynolds number appears in the generated axial pressure-gradient as well as the mean rate of mechanical work. The graph of variation of E with respect to r shows similar pattern as velocity profiles as shown in previous studies. This confirms the earlier assertion about the effect of the peristaltic motion.

Keywords: Peristalsis, Urine, Ureter, Pressure-Gradient, Efficiency.

Introduction

In an attempt to answer the question of how efficient our body performs its normal biological functions, Liron (1976), in a study titled *On Peristaltic Flow and Its Efficiency*, examines the case of peristaltic fluid transport and its efficiency in which complete solutions of equations for peristaltic flow of fluid, both for the case of a pipe and a channel, assuming a given time mean flow, are presented. These analytic solutions are obtained by a double expansion in terms of the Reynolds number and the square of the wave number. In the final determination, it is found that large plows are best for mechanical efficiency considerations, but large nipples use the least energy. Even though his study is premised on the previous works of Zien and Ostrach (1970) and subsequently Li (1970), there was no further mention of application to biological systems as proffered by these earlier works. Hence the interest of this study in application to urine flows.

A ureter is one of the two ducts (flow-tracts) that transmit urine from each kidney to the urinary bladder. Each duct emerges from each kidney, descends the abdominal cavity, and opens into the bladder. At its termination, the ureter passes through the bladder wall in such a way that, if the bladder fills, the terminal part of the ureter tends to close.

"The ureters (a pair) run obliquely through the muscular wall of the bladder for nearly 2cm before opening into the bladder cavity through narrow apertures. This oblique course provides a kind of valvular mechanism; when the bladder becomes distended, it presses against the part of each ureter that is in the muscular wall of the bladder, and this helps to prevent the flow of urine back into the ureter from the bladder. A ureter has thick contractile walls, and its cross-sectional area varies considerably at different points along its length, which is about $30cm$.

The main propelling force for the flow of urine from the kidney to the bladder is produced by the peristaltic movements in the ureter muscles."

Peristalsis is a form of fluid flow/transport mechanism achieved with the aid of a progressive contraction – expansion wavelike situation along the walls of the fluid-containing tract, which is assumed to be either rectangular (channel-like) or cylindrical (pipe/tube-like), depending on the location of the physiological situation under consideration. This mechanism is known to be one of the major fluid transport processes in many biological systems.

It occurs as involuntary movements of the longitudinal and circular muscles, primarily in the digestive tract, but occasionally in other hollow tubes of the body, that occur in progressive wavelike contractions. The waves can be short local reflexes, or long continuous contractions that travel the whole length of the organ, depending upon their location and what initiates their action.

Thus, peristaltic transport mechanism is involved in urine transport from kidney to bladder through the ureter.

Methodology

Though based on the Liron's (1976) study, in this case however, a specific organ is being considered, therefore a sinusoidal wave is strictly adopted as against arbitrary wave-shapes for the peristalsis. Also, based on other related previous studies by Zien and Ostrach (1970); and Li (1970), some major assumptions need to be made. These are:-

- The urine as produced by the kidneys and transported to the bladder is considered as an ideal Newtonian fluid.

- The ureter can be assumed to be tube-like (but not necessarily uniform), with axi-symmetrical flow.

- Flow of urine through the ureter is assumed to be steady and laminar.

- It is assumed that the peristaltic wave length is large compared to the radius of the ureter aperture. This assumption is to ensure laminar steady flow.

- The wave propagation speed is moderate so as to have low Reynolds number, (another situation or condition of steady flow).

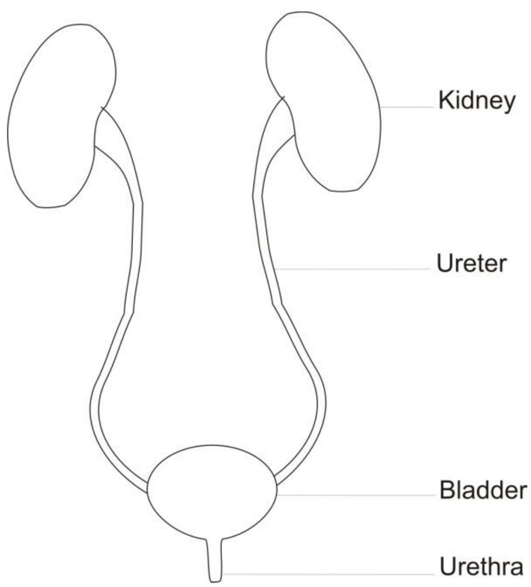

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Fig. 2: Model of the Upper Urinary System

2. 1 Problem Formulation

It considered the flow of a viscous incompressible fluid with density ρ and kinematic viscosity ν, moving in axi-symmetric form in a pipe or tube, under the influence of an infinite wave train, moving with velocity c along the walls. A two dimensional cylindrical coordinate system (R, Z) is chosen, with Z along the center line and R traverse to it. The pipe radius is taken as *d*.

$$R = \eta^{*}\left(Z, T\right), \text{ such that } \eta^{*} = d + a\sin\frac{2\pi}{\lambda}\left(Z - cT\right) \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Where: a is the amplitude,

λ is the wavelength,

T is time, while the motion of the wall is describe by R such that R is at η^*,

η^* is a differentiable function.

It is also assumed that the walls have only transverse motion, where U is the axial velocity, V is the transverse velocity. Thus boundary conditions will be:

$$U = 0;\ V = \frac{d\eta^{*}}{dT}, \text{ on the walls} \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

The Geometry of the model is shown below.

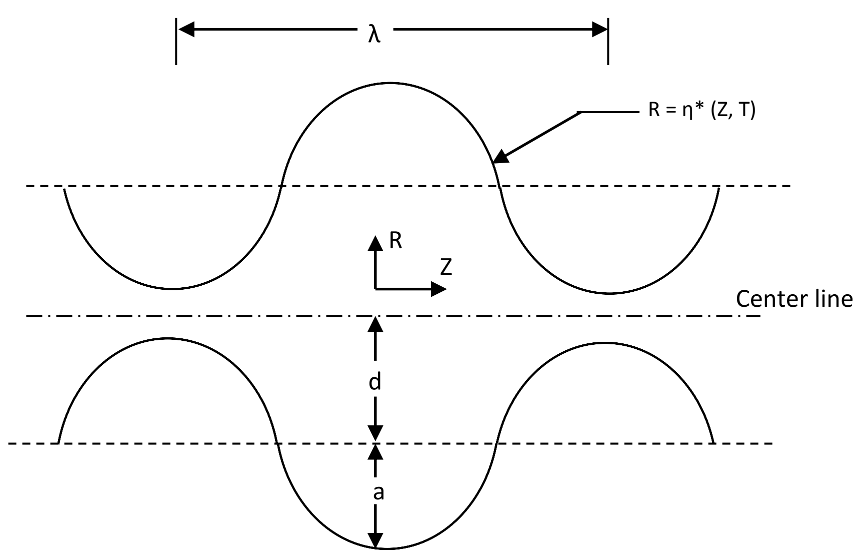


Fig. 3: A Two-Dimensional Pipe under peristaltic wave

Using the stream function method also, the Navier-Stokes equations of motion will now become:

$$D^2\Psi_T + \frac{1}{R}\Psi_Z\left[D^2\Psi_R - \frac{2}{R}D^2\Psi + \frac{1}{R^2}\Psi_R\right] - \frac{1}{R}\Psi_R D^2\Psi_Z = \nu D^4\Psi \qquad \ldots(3)$$

$$D^2 = \frac{\partial^2}{\partial Z^2} + \frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R},$$ the subscripts denote differentiation.

The velocities are given by:

$$U = -\frac{1}{R}\Psi_R \quad V = \frac{1}{R}\Psi_Z. \qquad \ldots(4)$$

The non-dimensional formulations give:

$$\left.\begin{aligned} & r = \frac{R}{d},\ z = \frac{Z}{\lambda},\ t = \frac{cT}{\lambda}, \\ & \psi = \frac{\Psi}{d^2 c},\ u = \frac{U}{c},\ v = \frac{V}{c},\ \eta = \frac{\eta^*}{d}. \end{aligned}\right\}. \qquad \ldots(5)$$

Substituting into (3) gives:

$$D^2\psi_t + \frac{1}{r}\psi_z\left(D^2\psi_r - \frac{2}{r}D^2\psi + \frac{1}{r^2}\psi_r\right) - \frac{1}{r}\psi_r D^2\psi_z = \frac{1}{Re}D^4\psi, \qquad \ldots(6)$$

where $$D^2 = \alpha^2\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}. \qquad \ldots(7)$$

With boundary conditions:

$$\frac{\partial\psi}{\partial r} = 0, \quad \frac{\partial\psi}{\partial z} = r\frac{\partial\eta}{\partial t} = -2\pi\varepsilon r\sin\left[2\pi\left(z-t\right)\right], \text{ (on } r=\eta) \qquad \ldots(8)$$

The three new dimensionless quantities that appear are the amplitude ratio $\varepsilon = \frac{a}{d}$, the wave number $\alpha = \frac{d}{\lambda}$ and the Reynolds number $Re = \left(cd/\nu\right)\alpha$.

2. 2 Solutions to Flow Equations

Going through the earliest studies in this field, the most popular or common analytical method for tackling these equations is the perturbation method, from which closed form solutions are derived. This is done by considering asymptotic expansions of the stream function ψ in terms of the wave number, α and the Reynolds number, *Re*.

Hence, using the same method as expounded by Liron (1976), here again, the expansion to be used is in the form:

$$\psi \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\psi_{i,j}\alpha^{2i}\,\mathrm{Re}^{i} \; . \qquad \ldots(9)$$

By substituting the expansion (9) into equation (6), the following partial differential equations recurrence relations for $\psi_{i,j}$ are obtained.

$$D_{\tau}^{2}\psi_{i.j} = -2\frac{\partial^2}{\partial z^2}D_r\psi_{i-1,j} - \frac{\partial^4}{\partial z^4}\psi_{i-2,j} + \frac{\partial^3\psi_{i,j-1}}{\partial z^2\partial t} + \frac{\partial}{\partial t}D_r\psi_{i,j-1}$$
$$+\frac{1}{r}\sum_{m=0}^{i}\sum_{n=0}^{j}\left\{\frac{\partial\psi_{m,n}}{\partial z}\left(\frac{\partial}{\partial r}-\frac{2}{r}\right)\left[\frac{\partial^2\psi_{i-m-1,j-n-1}}{\partial z^2}+D_r\psi_{i-m,j-n-1}\right]\right.$$
$$\left.-\frac{\partial\psi_{m,n}}{\partial r}\frac{\partial}{\partial z}\left[\frac{\partial^2\psi_{i-m-1,j-n-1}}{\partial z^2}+D_r\psi_{i-m,j-n-1}\right]\right\} \qquad \ldots(10)$$

$$\left.\begin{array}{l} i=0,1,2,\ldots,\quad j=0,1,2,\ldots,\\ \psi_{i,j}=0 \text{ if } i<0 \text{ or } j<0,\\ \text{and } D_r=\dfrac{\partial^2}{\partial r^2}-\dfrac{1}{r}\dfrac{\partial}{\partial r}, \end{array}\right\} \qquad \ldots(11)$$

With boundary conditions as:

$$\left.\begin{array}{l} \dfrac{\partial\psi_{0,0}}{\partial r}=0,\ \dfrac{\partial\psi_{0,0}}{\partial z}=r\dfrac{\partial\eta}{\partial t}=r\dfrac{\partial\eta}{\partial z},\ \text{on } r=\eta, \qquad (i)\\ \text{and } Q_0=\int_0^1 -2\pi\psi_{0,0}\,|_{r=\eta}\,dt\ \left(Q_0 \text{ is the time mean flow}\right)(ii)\\ \text{while for } i+j>0;\\ \qquad \dfrac{\partial\psi_{i,j}}{\partial r}=\dfrac{\partial\psi_{i,j}}{\partial z}=\psi_{i,j}=0,\ \text{on } r=\eta.\ \ \text{(iii)} \end{array}\right\} \qquad \ldots(12)$$

For the zeroth-order solution $\psi_{0,0}$ which is usually considered to be more "practically applicable" , the equation will be $D_r^2\ \psi_{0,0}\ = 0$. …(13)

The finite solution for which $r = 0$ is given by:

$$\psi_{0,0} = A_1 r^4 + A_2 r^2 \quad \text{…(14)}$$

For A_1, A_2 to satisfy the boundary condition (12)(i) will imply:

$$\eta^3 \frac{\partial A_1}{\partial z} + \eta \frac{\partial A_2}{\partial z} = -\frac{\partial \eta}{\partial z} \quad and \quad 2\eta^2 A_1 + A_2 = 0 \quad \text{…(15)}$$

By solving simultaneous gives:

$$A_1 = \left(C + \frac{1}{2}\eta^2 \right)\eta^{-4}\ ; \quad A_2 = -(2C\eta^{-2} + 1) \quad \text{…(16)}$$

Recall that, in this study, the case of purely natural peristaltic conditions is strictly being considered, with no externally imposed pressure gradient in whatever form. Thus as per the argument in Muhammad and Sesay, (2012):

$\Rightarrow \quad C = 0$; hence $A_1 = \frac{\eta^{-2}}{2}; \quad A_2 = -1$. Substituting into (14) gives:

$$\psi_{0,0} = \frac{1}{2}\eta^{-2} r^4 - r^2\ ;\ or\ \ \psi_{0,0} = \left(\frac{1}{2}\eta^{-2} r^2 - 1 \right) r^2 \quad \text{…(17)}$$

Where $\eta = 1 + \varepsilon \cos 2\pi (z - t)$.

Having obtained $\psi_{0,0}$ as above, then the other flow quantities can be obtained as:

Axial flow velocity:

$$v_{0,0} = -\frac{1}{r}\frac{\partial \psi_{0,0}}{\partial r} = -\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{2}\eta^{-2} r^4 - r^2 \right] = 2\left(1 - \eta^{-2} r^2\right) \quad \text{…(18)}$$

Mean volume flux:

$$Q_{0,0} = 2\pi \psi_{0,0}\Big|_{r=\eta} = \pi \eta^2 \quad \text{…(19)}$$

Generated axial Pressure gradient:

$$\frac{\partial p_{0,0}}{\partial z} = \frac{16}{Re\eta^4}\psi_{0,0} = \frac{16}{Re}\eta^{-4} r^2 \left(\eta^{-2} r^2 - 1 \right) \quad \text{…(20)}$$

It should be carefully noted that Liron was able to show that if the axial velocity is a periodic function of $(z-t)$, then in general, the pressure rise per wavelength is constant over a cross section.

3. 0 Work and efficiency

In this study also, the mechanical efficiency of the pumping process will be calculated, based on Shapiro and Jaffrin, (1969) definition as:

$$Efficiency = \frac{Mean\,Volume\,Flux \times Pressure\,Rise\,Per\,Wavelength}{Mean\,Rate\,of\,\,Mechanical\,Work}$$

$$\Rightarrow \quad E = \frac{Q \cdot \Delta P}{\overline{W}} \qquad \text{. . . (21)}$$

Considering the zeroth-order quantities, expressions for Q and ΔP are given in equations (19) and (20) respectively.

By definition: $\overline{W} = \iint_S \mathbf{t} \cdot \mathbf{v} dS$; where S is the surface of the wall of length λ in the axial direction of the ureter, **v** is the velocity vector, **t** is the stress vector.

It has been indicated earlier that in this case a situation of 'pure' natural peristalsis, with no any form of externally applied pressure gradient is envisaged. Thus considering the symmetry and laminar flow situations, the mean rate of mechanical work can be obtained as:

$$\overline{W} = \frac{2\pi r^2 c^3 \rho}{Re} \qquad \text{...(22)}$$

Substituting (19), (20) and (22) into (21) gives:

$$E = \frac{\pi\eta^2 \cdot \frac{16}{Re}\eta^{-4} r^2 \left(\eta^{-2} r^2 - 1\right)}{\frac{2\pi d^2 c^3 \rho}{Re}} = \frac{8\left(\eta^{-4} r^4 - \eta^{-2} r^2\right)}{d^2 c^3 \rho} \qquad \text{... (23)}$$

3. 1 Results and Discussions

It obvious that in this study, even though in the expressions for the 'generated' pressure gradient and the mean rate of mechanical work, appears the Reynolds Number factor, on the other hand, in the Efficiency factor what counts are the dimensions (radius *d*) of the flow tract; the nature (density ρ) of the fluid material; and the 'nature' (wave speed *c*, amplitude, etc.) of the peristalsis. To be able to analyse this quantity, (23) can be expressed as:

$$E = \frac{8\left(\eta^{-4}r^{4} - \eta^{-2}r^{2}\right)}{d^{2}c^{3}\rho} = \frac{8}{d^{2}c^{3}\rho}\left(\eta^{-4}r^{4} - \eta^{-2}r^{2}\right) \quad \ldots(24)$$

Since it is a study concerned with urine flow in the ureter, the various known properties of the ureter, which are of constant values, can be inserted, while the variation of *E* with respect to *r* will be analysed such as $0 < r < \eta$

It is pertinent to note that *r* cannot be zero i.e. $r \neq 0$, For $r = 0$ would imply a blockage of the flow tract which would mean there will be no flow condition (a clinical case for a ureter). So also, $r \neq \eta$. For $r = \eta$ would imply that there is no peristalsis, which would mean there will be no flow condition under peristalsis.

Let us first highlight some of the characteristics / properties of this organ(s) i.e. the ureter(s), as obtained from some earlier and later known physiological data (Weinberg, 1967; Encyclopedia Britannica – Ultimate Reference Suite, 2010);

- (i) Average Length = $0.30m$
- (ii) Average cross-sectional area (the hollow portion) = $7.857 \times 10^{-5} m^{2}$ (Diameter $4 - 5 \times 10^{-2} m$).
- (iii) Total Peristaltic waves (from beginning to end) – 3 to 4 times.
- (iv) Average Wave Amplitude – $0.005m$ average
- (v) Wave Speed – $0.03ms^{-1}$ to $0.06ms^{-1}$
- (vi) Wave Frequency – 1 to 8 per min.
- (vii) Contraction – lasts $1.5secs.$ to $9secs.$
- (viii) Dilating Phase is about twice as long as the contracting phase.
- (ix) Generated Pressure during contraction is:-
  - At the pelvis - $2mmHg$ to $8mmHg$,
  - At the upper segment of ureter - $2mmHg$ to $10mmHg$,

- At the lower segment of ureter - $2mmHg$ to $14mmHg$.

From the above information one can assume the average values for the following $\bar{d} = 4\cdot5\times10^{-2}$; $\bar{c} = 4\cdot5\times10^{-2}$; $\rho = 1$; $5\cdot0\times10^{-3} \le r \le 4\cdot5\times10^{-2}$.

Thus equation (24) becomes:

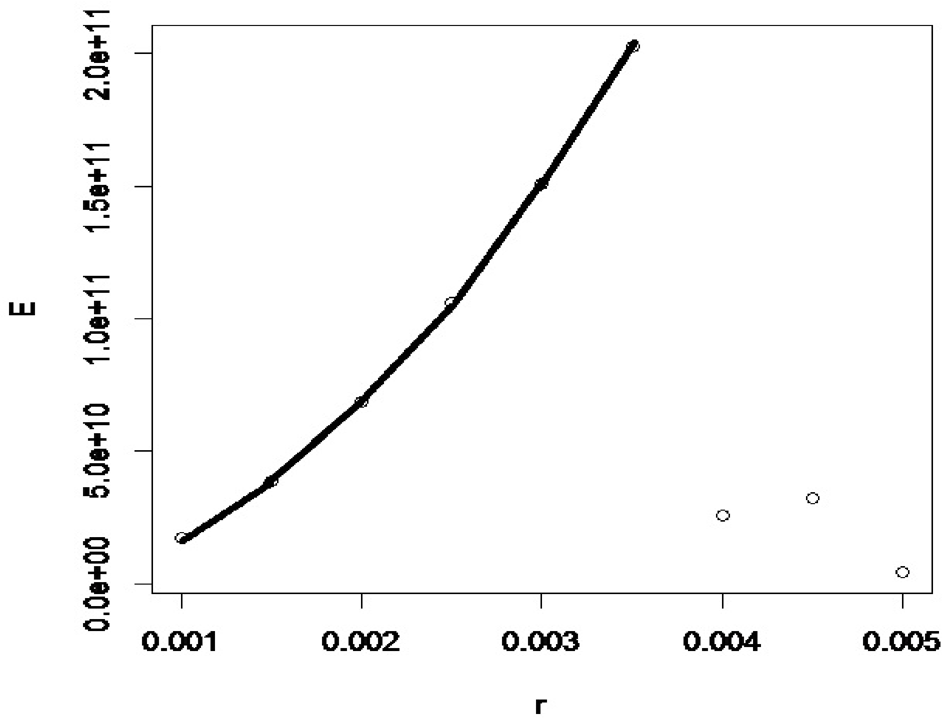
$$E = 4.32\times10^{7}\left(\eta^{-4}r^{4} - \eta^{-2}r^{2}\right). \qquad \ldots(25)$$

Substituting η by the maximum wave amplitude $\varepsilon = 5\cdot0\times10^{-3}$, then equation (25) can be modified to:

$$E = 1.73\times10^{14}\left(40r^{4} - r^{2}\right). \qquad \ldots(26)$$

Drawing the efficiency profile as *r* varies from flow tract centre to the walls in form of a graph will show the shape below:

r	5×10^{-3}	10×10^{-3}	15×10^{-3}	20×10^{-3}	25×10^{-3}	30×10^{-3}	35×10^{-3}	40×10^{-3}	45×10^{-3}
E-	4.35×10^{9}	17.40×10^{9}	38.80×10^{9}	68.56×10^{9}	105.97×10^{9}	151.03×10^{9}	202.71×10^{9}	260.65×10^{9}	323.81×10^{9}



The above graph shows similar pattern as in the case of velocity profiles across the cross-section of a tube under peristalsis shown in the previous studies. This confirms the assertion by this study that the efficiency of the peristaltic pumping may likely depend more on the wave nature of the peristalsis rather than the particular fluid in the case of physiological situations.

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