NUMERICAL TREATMENT OF HIGHER ORDER DIFFERENTIAL EQUATIONS USING RUNGE KUTTA AND PREDICTOR CORRECTOR METHODS

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ABSTRACT

In this study, two numerical methods were used to obtain a numerical solution of higher order ordinary differential equations. The two methods are the existing Runge Kutta method and Adam-Bashforth Moulton predictor corrector methods which have been modified to accommodate the general n^{th} order ordinary differential equations. Four higher order differential equation problems were solved and the results obtained were consistent with when compared with the exact solution.

Keywords: Runge Kutta, Predictor, Corrector, Absolute Error.

INTRODUCTION

Higher order ordinary differential equations are expressions that involve derivatives other than the first. Differential equations arise in many areas of science and technology specifically whenever a deterministic relation involving some continuously varying quantities and their rate of change in the space and time (expressed as derivatives) is knownor postulated. This is illustrated in classical mechanics, where the motion of a body is described by its position and velocity as the time varies. Newton's laws allow one to relate the position, velocity, acceleration and various forces acting on a body and states the relation as a differential equation for unknown position of the body as a function of time. The study of differential equations is wide field in pure and applied mathematics, physics, meteorology and engineering. All of these disciplines are concern with the properties of differential equations of various types. Pure mathematics focuses on the existence and uniqueness of solutions while applied mathematics emphasizes the rigorous justification of the methods for approximating solutions (Butcher, 1999). Many problems in science and engineering can be reduced to the problem of solving differential equations under certain conditions. The analytical methods of solution can be applied to solve only a selected class of differential equations. Those equations which govern physical systems do not process in general closed form solutions and hence recourse must be made to numerical methods for solving such differential equations (Butcher, 2003).

RUNGE KUTTA METHOD

This method was devised by two German mathematicians, Runge about 1894 and extended by Kutta a few years later. It is powerful tool for the solution of ordinary differential equation (ODE). Most of the research has been oriented towards improving the accuracy orthe flexibility (to accommodate problems of diverse nature)

(Goeken et al., 1999). The most widely member of the Runge Kutta family is generally referred to as RK4 Classical Runge Kutta method or simply as the Runge Kutta method (Goeken et al., 1999). Let an initial value problem be specified as follows;

$$\frac{dy}{dx} = f(x, y), \qquad y = y_0 \ at \ x = x_0 \quad (1)$$

Here y is an unknown function (scalar or vector) of time x which we would liketo approximate; we are told that $\frac{dy}{dx}$ the rate at which y changes is a function of x and of y itself. At the initial time x_0 the corresponding y value is y_0. The function f and the data x_0, y_0 are given. Now pick a step size $h > 0$ and define

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \quad (2)$$

$$x_{n+1} = x_n + h$$

For $n = 0,1,2,...$ using

$$k_1 = hf(x_n, y_n),$$

$$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$

$$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$$

$$k_4 = hf(x_n + h, y_n + k_n)$$

Equation (2) is the RK4 approximation of $y(x_{n+1})$, and the next value of (y_{n+1}) is determined by the present value (y_n) plus the weighted average of four increment, where each increment is the product of the size of the interval h and an estimated slope specified by function f on the right hand side of the differential equation.

For higher order differential equation, we introduce scalar variables (i, j) as shown below y_{n+1} as $y_{i,j+1}$, $i = 0,1,2,3,...$ (depends on the equation), $j = 0,1,2,...,n$

$$y_{i,j+1} = y_{i,j} + \frac{1}{6}(k_{1,i} + 2k_{2,i} + 2k_{3,i} + k_{4.i}) \quad (3)$$

$$k_{1,i} = hf_i(x_j, y_{1,j}, y_{2,j}, y_{3,j}...)$$

$$k_{2,i} = hf_i(x_j + \frac{h}{2}, y_{1,j} + \frac{k_{1,1}}{2}, y_{2,j} + \frac{k_{1,2}}{2}, y_{3,j} + \frac{k_{1,3}}{2})$$

$$k_{3,i} = hf_i(x_j + \frac{h}{2}, y_{1,j} + \frac{k_{2,1}}{2}, y_{2,j} + \frac{k_{2,2}}{2}, y_{3,j} + \frac{k_{2,3}}{2})$$

$$k_{4,i} = hf_i(x_j + h, y_{1,j} + k_{3,1}, y_{2,3} + k_{3,2}, y_{3,j} + k_{3,3}$$

Adams Predictor-Corrector Method

Predictor-corrector methods refer to a family of schemes for solving ordinary differential equations using two formulae; predictor and corrector formulae. In predictor-corrector methods four prior values are required to find the value of y at x_n. We consider a differential equation (1). This problem is to be solved by Adam-Bashforth-Moulton method. It is a fourth step predictor-corrector method. So, to compute the value of y_{i+1}, four values $y_{i-3}, y_{i-2}.y_{i-1}$ and y_n are required. These values are called starting values. Generally, any single step methods such as Euler, Runge-Kutta, etc,are used to find these values. Now, we integrate the differential equation (1) between x_i and x_{i+1} and obtain the following equation

$$y_{n+1} = y_i + \int_{x_i}^{x_{i+1}} f(x, y)dx \quad (4)$$

The second term of the above equation cannot be determined as y is an unknown dependent variable to find this integration, the function $f(x, y)$ is approximate by Newton's backward interpolation formula below;

$$f = f_i + v\nabla f_i + \frac{v(v+1)}{2!}\nabla^2 f_i + \frac{v(v+1)(v+2)}{3!}\nabla^3 f_i \qquad (5)$$

Where $v = x - x_i h$ and $f_i = f(x_i, y_i)$. After simplification, it becomes

$$f = f_i + vf_i + \frac{v^2+v}{2}\nabla^2 f_i + \frac{v^3+3v^2+2v}{6}\nabla^3 f_i$$

Therefore,

$$y_{i+1} = y_i + h\int_0^1 [f_i + v\nabla f_i + \frac{v^2 + v\nabla^2 f_i}{2} + \frac{v^3+3v^2+2v}{6}\nabla^3 f_i]dv$$

$$= y_i + hf_i + \frac{1}{2}h\nabla f_i + \frac{5}{12}h\nabla^2 f_i + \frac{3}{8}\nabla^3 f_i \qquad (6)$$

$$= y_i + \frac{h}{24}(-9f_{i-3} + 37f_{i=2} - 59f_{i-1} + 55f_i\ )$$

This formula is known as Adam Bashforth predictor formula and it is denoted by y_{i+1}^p

Thus,

$$y_{i+1}^p = y_i + \frac{h}{24}[-9f(x_{i-3}, y_{i-3}) + 37f(x_{i=2}, y_{i-2}) - 50f(x_{i-1}, y_{i-1}) + 55f(x_i, y_i) \qquad (7)$$

To obtain the corrector formula, the function $f(x.y)$ is approximated by the following Newton's backward interpolation polynomial defined in equation (5)

$$f(x,y) = f_{i+1} + v\nabla f_{i+1} + \frac{v(v+1)}{2!}\nabla^2 f_{i+1} + \frac{v(v+1)(v+2)}{3!}\nabla^3 f_{i+1}$$

where $v = x - x_{i+1}h$ using this approximation, the equation (2) becomes,

$$y_{i+1} = y_i + h\int_{-1}^0 [f_{i+1} + v\nabla f_{i+1} + \frac{v^2+v}{2}\nabla^2 f_{i+1} + \frac{v^3+3v^2+2v}{6}\nabla^3 f_{i+1}]dv$$

Since, $x = x_n + vh, dx = hdv$

$$= y_i + h[f_{i+1} - \frac{1}{2}\nabla f_{i+1} - \frac{1}{12}\nabla^2 f_{i+1} - \frac{1}{21}\nabla^3 f_{i+1}]$$

$$= y_i + \frac{h}{24}[f_{i-2} - 5f_{i-1} + 19f_i + 9f_{i+1}]$$

This is known as Adams Moulton corrector formula. The corrector value is denoted by y_{i+1}^c.

Thus,

$$y_{i+1}^c = y_i + \frac{h}{24}[f(x_{i=2}, y_{i-2}) - 5f(x_{i-1}, y_{i-1}) + 19f(x_i, y_i) + 9f(x_{i+1}, y_{i+1}^p)] \qquad (8)$$

The predicted value y_{i+1}^p is computed from equation (6). The formula (8) can be used repeatedly to get the value of y_{i+1} to the desired accuracy. "Note that the predictor formula is an explicit formula, whereas, corrector formula is an implicit one".

The derived Adam Bashforth Predictor-Corrector Method can be used to solve higher order differential equation as shown below;

$y_{i+1}, y_{i,j+1}, i = 1,2,3,\ldots$ (Depends on the order of the equation), $j = 3,4,5,\ldots,n$

$$y^{p}_{i,j+1} = y_{i,j} + \frac{h}{24}[-9f_{j-3,i} + 37f_{j-2,i} - 59f_{j-1,i} + 55f_{j,i}] \quad (9)$$

$$y^{c}_{i+1} = y_i + \frac{h}{24}[f_{j=2,i} - 5f_{j-1,i} + 19f_{j,i} + 9f_{j+1,i}] \quad (10)$$

Numerical Problems

Problem 1: $y'''(x) + 2y''(x) - y'(x) - 2y(x) = e^x$

The exact solution $y(x) = \frac{43}{36}e^x + \frac{1}{4}e^{=x} - \frac{4}{9}e^{-2x} + \frac{1}{6}xe^x$

$$y(0) = 1, y'(0) = 2, y''(0) = 0, h = 0.2$$

Solution;

Reducing the equation to a system of first order ordinary differential equation

let $y(x) = y_1(x)$

$$y_1'''(x) + 2y_1''(x) - y_1'(x) - 2y_1(x) = e^x$$

let $y_1'(x) = y_2(x)$

$$y_2''(x) + 2y_2''(x) + y_2(x) - 2y_1(x) = e^x$$

let $y_2'(x) = y_3(x)$

$$y_3'(x) + 2y_3(x) - y_2(x) - 2y_1(x) = e^x$$

$$y_3'(x) = e^x - 2y_1(x) + y_2(x) + 2y_3(x)$$

$$y_3'(x) = e^x - 2y_{1,0} + y_{2,0} - 2y_{3,0}$$

Thus, the order reduction above generated three system of first order ordinary differential equation which are

$$y_1'(x) = y_2(x)$$

$$y_2'(x) = y_3(x)$$

$$y_3'(x) = e^x - 2y_1(x) + y_2(x) + 2y_3(x)$$

Table 1: Calculated Error from Runge Kutta Method for Problem (1)

X	Exact solution	j	$y_{i,j}$		
			$i=1$	$i=2$	$i=3$
0.0	1	0	0	1	1
0.2	1.406373832	1	3.705200 x 10^{-3}	6.88021552 x 10^{-1}	4.86768273 x 10^{-1}
0.4	1.849234952	2	5.349800 x 10^{-3}	5.12654302 x 10^{-1}	1.02005638 x 10^{-1}
0.6	2.361970373	3	6.134200 x 10^{-3}	4.30930264 x 10^{-1}	2.05606482 x 10^{-1}
0.8	2.977624244	4	6.781400 x 10^{-3}	4.15482851 x 10^{-1}	1.479603948
1.0	3.731704445	5	7.266600 x 10^{-3}	4.49544655 x 10^{-1}	1.918635487

Table 2: Calculated Error from Predictor Corrector Method for Problem (1)

X	Exact solution	j	$y_{i,j}$		
			$i=1$	$i=2$	$i=3$
0.0	1	0			
0.2	1.406373832	1			
0.4	1.849234952	2			
0.6	2.361970373	3	0	4.30872931 x 10^{-1}	2.05780651 x 10^{-1}
0.8	2.977624244	4	7.0942 x 10^{-5}	4.15402622 x 10^{-1}	4.72851249 x 10^{-1}
1.0	3.731704445	5	4.8767 x 10^{-5}	3.81576307 x 10^{-1}	8.82826514 x 10^{-1}

Problem 2: $y'''(x)+3y''(x)+y'(x)+3y(x)=\sin(2x)$

The exact solution:

$$y(x)=\frac{2}{125}\cos(2x)-\frac{11}{125}\sin(2x)-\frac{2}{125}e^{-x}+\frac{29}{25}e^{-x}x+\frac{11}{5}e^{-x}x^2$$

$$y(0)=0, y'(0)=1, y''(0)=2, h=0.2$$

Table 3: Calculated Error from Runge Kutta Method for Problem (2)

X	Exact solution	j	$y_{i,j}$		
			$i=1$	$i=2$	$i=3$
0.0	0	0	0	-1	-2
0.2	0.264279408	1	3.2546075 x 10^{-2}	1.013943863	6.03366705 x 10^{-1}
0.4	0.550125805	2	5.1787011 x 10^{-2}	8.19693108 x 10^{-1}	4.44637524 x 10^{-1}
0.6	0.822004286	3	5.0369977 x 10^{-2}	5.12762064 x 10^{-1}	1.258230923
0.8	1.055979854	4	2.7981739 x 10^{-2}	1.49266054 x 10^{-1}	1,896853485
1.0	1.243107949	5	6.968682 x 10^{-3}	2.37720363 x 10^{-1}	1.00543850 x 10^{-1}

Table 4: Calculated Error from Predictor Corrector Method for Problem (2)

X	Exact solution	j	$y_{i,j}$		
			$i=1$	$i=2$	$i=3$
0.0	0.0	0			
0.2	0.264279408	1			
0.4	0.551025805	2			
0.6	0.822004286	3	0	3.57623853 x 10^{-1}	2.599691809
0.8	1.055979854	4	1.50271077 x 10^{-1}	1.036860426	2.405324793
1.0	1.243107949	5	1.29349401 x 10^{-1}	1.223988521	2.730148588

Problem 3: $y''(x)+6y'(x)+9y(x)=5e^{3x}$

The exact solution $y(x)=(\frac{31}{36}+\frac{13x}{6})e^{-3x}+\frac{5}{36}e^{3x}$

$y(0)=1, y'(0)=0, h=0.1$

Table 5: Calculated Error from the two Method for problem (3)

X	Exact solution	j	$y_{i,j}$		$Y_{i,j}$	
			i=1	i=2	I=1	i=2
0	1	0	0	1		
0.1	0.985917805	1	7.5153×10^{-5}	1.235913166		
0.2	0.963478229	2	3.689079×10^{-3}	1.120140458		
0.3	0.955983473	3	5.430519×10^{-2}	$8.81924657 \times 10^{-1}$	0	$8.93363705 \times 10^{-1}$
0.4	0.981524016	4	6.004298×10^{-3}	$5.04111544 \times 10^{-1}$	2.5639113×10^{-2}	9.4962163×10^{-2}
0.5	1.056321016	5	7.2557242×10^{-2}	1.758115750	7.4935001×10^{-2}	1.141621638

Problem 4: $y''(x)=y'(x)$

The exact solution $y(x)=e^{x}$

$y(0)=1, y'(0)=1, h=0.1$

Table 6: Calculated Error from the Two Method for problem 4

X	Exact solution	j	$y_{i,j}$		$Y_{i,j}$	
			i=1	i=2	I=1	i=2
0	1	0	0	1		
0.1	1.105170918	1	1.70918×10^{-4}	5.170919×10^{-3}		
0.2	1.221402758	2	9.02758×10^{-4}	1.1402758×10^{-2}		
0.3	1.349858808	3	2.308808×10^{-3}	1.8858808×10^{-2}	0	0
0.4	1.491824698	4	4.519698×10^{-3}	2.7724698×10^{-2}	$1.41966590 \times 10^{-1}$	$1.41966590 \times 10^{-1}$
0.5	1.648721271	5	7.685771×10^{-3}	3.8211271×10^{-2}	$1.56897339 \times 10^{-1}$	$1.56897339 \times 10^{-1}$

CONCLUSION

From the tables above, the Predictor-Corrector is more consistent than the Runge Kutta method for its value tends to the exact solution and it was discovered from our findings that it minimizes the computation time and number of iterations. We can observe from the table that as the order of the differential equation increases, the error increases as well, this is due to the fact that these methods were meant to handle only first order ordinary equations.

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