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INTRODUCTORY p -BANACH ALGEBRAS

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ABSTRACT

The researchers identified and defined p -normed algebra and p -Banach algebra on certain quasi-normed spaces such as the space of bounded linear operators $\left(L(X,Y),\|\cdot\|_{L(X,Y)}\right)$ defined on quasi-normed spaces $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ and the space $L^0(\mu,C)$ of μ -measurable complex-valued functions defined on p -Banach space $\left(X,\|\cdot\|_p\right)$, $0<p\leq 1$.

Key words: quasi-normed space, p -normed space, p -normed algebra, quasi-Banach space, p -Banach space and p -Banach algebra.

INTRODUCTION

Banach spaces X as we know are complete normed spaces. Complete in a sense of the metric $d(x,y)=\|x-y\|,\ \forall x,y\in X$ defined by the norm. They are spaces that are rich in functionals, thus one of the famous theorems in functional analysis: the Hahn-Banach theorem and thus Banach algebra could be used to analyze them. However, there are some spaces which do not meet the Banach space conditions, the triangle inequality property of the norm, instead they have flipped triangle property of the norm. These categories of spaces are generally non-locally convex and are not rich in functionals. And thus, the algebra used in their analysis is called the p -Banach algebra. We shall begin by briefly looking at the following basic concepts.

Quasi-Normed Space

By definition, a quasi-norm on a vector space X is real-valued

$$\|\cdot\|: X\to[0,+\infty],$$

such that for every $x,y\in X$ and all scalar α, the following axioms are satisfied:

$$\|x\|\geq 0 \text{, and } \|x\|=0 \text{ if, and only if } x=0. \tag{1.1}$$

$$\|\alpha x\|=|\alpha|\|x\| \tag{1.2}$$

$$\|x+y\|\leq K\left(\|x\|+\|y\|\right) \tag{1.3}$$

where the K in (1.3) is a positive constant that is independent of x or y. The least such constant K is called the modulus of concavity of $\|\cdot\|$. The space X together with the quasi-norm satisfying (1.1) to (1.3) is called a quasi-normed space and is denoted by $(X, \|\cdot\|)$ or $(X, \|\cdot\|_X)$ where $\|\cdot\|_X$ implies a quasi-norm with respect to the space X in a situation where more than one quasi-normed space is involved as in the case of a mapping between quasi-normed spaces.

A linear transformation $T : X \to Y$ between quasi-normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is defined by ;

$$\|T\|_Y = \sup\{\|T(x)\|_Y : x \in X \wedge \|x\|_X \leq 1\} \tag{1.4}$$

If we denote a space of the linear transformation defined in (1.4) by $L(X, Y)$ defined a quasi-normed on it by $\|\cdot\|_{L(X,Y)}$, then the space $(L(X,Y), \|\cdot\|_{L(X,Y)})$ of bounded linear operators defined $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is a quasi-normed space.

A quasi-normed X is said to be a p -normed space if in addition of properties (1.1) to (1.3), the following property is satisfied.

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p, \; 0 < p \leq 1, \; \forall x, y \in X, \tag{1.5}$$

in which case, the quasi-norm is replaced by a p -norm $\|\cdot\|_p$, $0 < p \leq 1$.

A complete p -normed space is called p -Banach space.

Examples of a p -Banach space include the space $L^0(\mu, C)$ of μ -measurable complex-valued functions defined on X -a quasi-Banach space, where for $f \in L^0(\mu, C)$, X is defined by;

$$X = \{f \in L^0(\mu, C) : \|f\| < \infty\} \tag{1.6}$$

Algebra

A vector space X over a field F is called an algebra (over F) if, and only if there is a multiplication in which X becomes also a ring. The property which relates the two structures is

$$\lambda(xy) = (\lambda x)y = x(\lambda y)$$

The multiplication is associative. In addition, there is a unit which is preserved under homomorphism.

An algebra X with a norm (over F) is a vector space with a norm in which X is an F -algebra too. The properties which relate the two structures are;

$$\|xy\| \leq \|x\|\|y\|, \; \forall x, y \in X \tag{2.1}$$

$$\|I\| = 1 \tag{2.2}$$

Property (2.1) makes multiplication to be jointly continuous in a normed algebra. Now, if in addition X is Banach space, then it is called a Banach algebra.

p -Banach Algebra

By definition, a p -normed algebra is a p -normed space $\left(X, \|\cdot\|_p\right)$, $0 < p \le 1$, which is also an algebra. The properties which relate the two structures are;

$$\|xy\|_p \le \|x\|_p \|y\|_p, \; 0 < p \le 1, \tag{2.3}$$

$$\|I\|_p = \|I\| = 1 \tag{2.4}$$

Now, if in addition $\left(X, \|\cdot\|_p\right)$, $0 < p \le 1$, is a p -Banach space, then $\left(X, \|\cdot\|_p\right)$, $0 < p \le 1$, is called a p -Banach algebra.

RESULTS

The study seek to establish that the space of bounded linear operator defined on quasi-normed spaces and that of μ -measurable complex-valued functions defined on X - a quasi-Banach space considered above are indeed p -normed algebra and p -Banach algebra respectively as presented in the following result.

Proposition 4.3. Let $(L(X,Y), \|\cdot\|_p)$, $0 < p \le 1$, be a p -normed space of bounded linear operators. Then for every $T, S \in (L(X,Y), \|\cdot\|_p)$, $0 < p \le 1$, $(L(X,Y), \|\cdot\|_p)$, $0 < p \le 1$, is a p -normed algebra.

Proof.

To verify that $(L(X,Y), \|\cdot\|_p)$, $0 < p \le 1$, is a p -normed algebra we work as follows. Let $T, S \in (L(X,Y), \|\cdot\|_p)$, $0 < p \le 1$, be pair of bounded linear operators, for every $x \in X$ we have;

$$\begin{aligned}\|TS(x)\|_Y^p &= \sup\left\{\|TS(x)\|_Y^p : \|x\|_X^p \le 1\right\}, \text{ by definition, see (2.16)} \\ &\le \sup\left\{\|T\|_Y^p \|S(x)\|_Y^p : \|x\|_X^p \le 1\right\}, \\ &\le \sup\left\{\|T\|_Y^p : \|x\|_X^p \le 1\right\} \sup\left\{\|S\|_Y^p : \|x\|_X^p \le 1\right\} \\ &= \|T\|_Y^p \|S\|_Y^p\end{aligned}$$

Hence, $\|TS\|_Y^p \le \|T\|_Y^p \|S\|_Y^p$.

Generally, $\|TS\|_p \le \|T\|_p \|S\|_p$, $0 < p \le 1$, showing that $(L(X,Y), \|\cdot\|_p)$, $0 < p \le 1$ is indeed a p -normed algebra. The unit element is the identity operator I.

To check this, we have since $\|Ix\|_p = \|x\|_p \le \|x\|_p$, then $\|I\|_p \le 1$ and since

$$\|x\|_p = \|Ix\|_p \le \|I\|_p \|x\|_p, \text{ then } 1 \le \|I\|_p, \text{ whence } \|I\|_p = 1.$$

The space of quasi-Banach functions discussed under section (2.4) is a p -Banach algebra if the quasi-norm is replaced by a p -norm. $\|\cdot\|_p$, $0 < p \le 1$. This is presented in the next result.

Proposition (4.4). Let $(X,\|\cdot\|_p)$, $0<p\le 1$, be p-Banach function space, then X is a p-Banach algebra.

Proof.

Let $f,g\in L^0(\mu,C)$ be μ-measurable functions defined on X, where X is a p-Banach space $(X,\|\cdot\|_p)$, $0<p\le 1$, then

$$\|fg\|_p = (\int_X |fg(x)|^p d\mu(x))^{\frac{1}{p}} \text{ (by definition)}$$
$$\le (\int_X (|f\|g|)^p d\mu)^{\frac{1}{p}}$$
$$\le (\int_X |f|^p|g|^p d\mu)^{\frac{1}{p}}$$
$$\le (\int_X |f|^p d\mu)^{\frac{1}{p}}(\int_X |g|^p d\mu)^{\frac{1}{p}}$$
$$= \|f\|_p\|g\|_p$$

Hence, $\|fg\|_p \le \|f\|_p\|g\|_p$

The unit is essentially the constant function.

Therefore, $(X,\|\cdot\|_p)$, $0<p\le 1$, is a p-Banach algebra.

CONCLUSION

In order to establish our claims, we started by considering some preliminaries which are essential especially for any researcher having interest in the subject area, but who has no basic background. In addition, the established results can be used for further studies on the properties of such spaces, such as the spectral values of these spaces.

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