URINATION AS A CASE OF HAGEN-POISEUILLE FLOW

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ABSTRACT

The flow of urine in the human urethra is modeled as a natural voluntary flow of incompressible Newtonian fluid in a cylindrical flow tract (tube or pipe-like), with a view to test the suitability of Hagn-Poiseulles flow theory to natural flow of fluids in human body. With the assumption of the cylindrical nature of the urethra and the voluntary nature of urine pumping, the urethral model is tried. The resultant flow quantities in terms of the axial flow velocity, volume flux, and generated axial pressure gradient, as obtained by solving the modified Navier-Stokes equations, in particular the flow speed, tend to agree with some experimental figures obtained in earlier studies. This shows the possible application of this type of flow to similar physiological situations.

Keywords: Urethra, Urine, Hagen-Poiseuille Flow, Cylindrical Tract, Stricture)

INTRODUCTION

In this study, the urination process is regarded as practically consisting of four stages, comprising of storage (in the bladder); voiding (function of the urethra); control situation – voluntary or involuntary (by the central nervous system); and the exit of the urine (the urethral flow tract). However, the main issue of concern here is the urethra and its urinary functions.

Position and Structure of the Urethra

Resembling a tubular structure, the urethra develops a connection between the urinary bladder and the genital organs in human body. It is this canal, through which the transfer of excretory fluids occurs, that is ultimately emptied out of the body.

The length of urethra shows considerable variation based on gender differences, i.e. in women, it ranges between 0.035*m* to 0.040*m*, while in males, it varies from 0.15*m* to 0.29*m*. This is because, in women, it travels a shorter distance to the outside from the internal to the external urethral orifice. In case of men, it has to cover comparatively long distance to reach the end of the male genital organ.

The urethral body wall is composed of three distinct layers that are continuous with the urinary bladder. However, to make the concept clear, the urethral tube can be divided into four distinct parts which are named after their respective location, namely, spongy, membranous, pre-prostatic and prostatic urethra".

Functions of Urethra

In women, the urethral tube serves only as a passage for urine from bladder to outside. For men however, it is involved in dual functions, the transfer of urine out of the body as well as the ejaculation of seminal fluid.

“The process of urination is controlled by both voluntary and involuntary actions where the external urethral sphincter is responsible for the deliberate command over the excretion of urinary fluid. The striated muscle tissue that forms the external voluntarily controlled sphincter is responsible for the effective functioning of this structure” (Wikepedia- 10/10/2017).

Urine and Urination (Urine Flow)

This is the flow (by voluntary or involuntary pumping) of urine from the bladder through the urethra to outside the body. It is also known medically as *micturition*, *voiding*, or *uresis*; and known colloquially by various names including *peeing, weeing*, or *pissing*. The need to urinate is experienced as an uncomfortable, full feeling. It is highly correlated with the fullness of the bladder. In many males the feeling of the need to urinate can be sensed at the base of the bladder, even though the neural activity associated with a full bladder comes from the bladder itself, and can be felt there as well. In females the need to urinate is felt in the lower abdomen region when the bladder is full. When the bladder becomes too full, the sphincter muscles will involuntarily relax, allowing urine to pass from the bladder. Release of urine is experienced as a lessening of the discomfort.

In normal situations (healthy conditions) urination is voluntary controlled, except in the case of infants and very elderly, where it comes under reflex (involuntary) situation. “The state of the reflex system is dependent on both a conscious signal from the brain and the firing rate of sensory fibers from the bladder and urethra. At low bladder volumes, afferent firing is low, resulting in excitation of the outlet (the sphincter and urethra), and relaxation of the bladder. At high bladder volumes, afferent firing increases, causing a conscious sensation of urinary urge”.

“The muscles controlling urination process are controlled by the autonomic and somatic nervous systems. During the storage phase the internal urethral sphincter remains tense and the detrusor muscle relaxed by sympathetic stimulation. During micturition, parasympathetic stimulation causes the detrusor muscle to contract and the internal urethral sphincter to relax. The external urethral sphincter (sphincter urethrae) is under somatic control and is consciously relaxed during micturition. It is commonly believed that in infants, voiding occurs involuntarily (as a reflex). The ability to voluntarily inhibit micturition develops by the age of 2–3 years, as control at higher levels of the central nervous system develops.

Hagen-Poiseuille Flow

This is the case of flow of a viscous fluid through a pipe or tube in one direction along the axis of symmetry. In this study a case of incompressible fluid under steady flow is being considered. The study is intended to analyze the situation of voluntary flow of fluid (urine) through the human (male) urethra, based on the Hagen-Poiseuille’s model with a view to study the case(s) of some of the clinical conditions that cause disturbances to normal urination. It is common to note that some males prefer to urinate standing while others prefer to urinate sitting or squatting. Elderly males and others with urinary problem may prefer sitting down, while in healthy males, no difference is found in the ability to urinate. Thus, the mathematical study of urine flow in the urethral tract with a view to determining some of the clinical conditions that cause disturbances to normal urination will be of great interest.

MATERIALS AND METHODS

Major Assumptions

Having assumed the urethra to be cylindrical based on the fact that "the cylindrical model will be more suitable for natural studies" (Muhammad and Sesay 2010), there are some further assumptions that need to be made here also, to enable one carry out this study. These are: -

(i) Urine as produced by the kidneys and transported to the bladder and finally expelled outside the body through the urethra is considered as an ideal Newtonian fluid.

(ii) The urethra can be assumed to be a cylindrical (tube-like) flow tract, with axi-symmetrical flow.

(iii) Flow of urine through the urethra is assumed to be steady and laminar (though may not necessarily be so).

Problem Formulation

From the above assumptions therefore, the two-dimensional Navier-Stokes equations and the continuity equation, the cylindrical model can be applied to describe the flow processes. In this regards it will be easier to consider a very small portion at the mid-section of the urethra, where the effect of tapering (if there is any) may be minimal. Thus, one can take the cross-section of the urethra at that portion and determine the behavior or variation of the quantities there. The geometry will be as shown below:

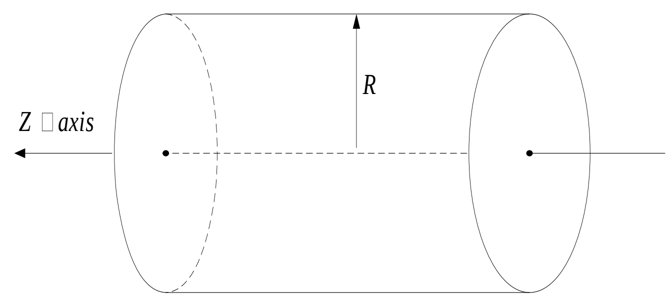

Figure 1: Diagram showing the Flow in a Straight Cylindrical Pipe or Tube.

The cylindrical coordinates system is applied since a cylindrical flow tract is assumed. Thus the continuity equation will be:

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(\rho r v_r\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho v_\theta\right)+\frac{\partial}{\partial z}\left(\rho v_z\right)=0. \quad \ldots(3.1)$$

Where ρ is the fluid density; μ the fluid viscosity; R the radius of the flow tract; *r* the perpendicular distance from the axis; v_z , v_r and v_θ are the axial, the radial and the swirl velocities respectively.

(There are no radial and swirl velocities, implying that: $v_r = v_\phi = 0$).

Since a case of steady flow along the Z-axis is being considered, the only term that will remain in equation (3.1), is $\frac{\partial v_z}{\partial z} = 0$; implying that v_z is not a function of z; hence $v_z = f(r)$.The appropriate N-S equation to be applied for the flow along the Z-axis will be (Muhammad 2017):

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right)$$

$$= -\frac{\partial p}{\partial z} + \mu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right\}, \qquad \ldots(3.2)$$

This becomes:

$$-\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] = 0; \text{ or } -\frac{\partial p}{\partial z} = \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]. \qquad \ldots(3.3)$$

Because it is assumed that there is no variation in pressure gradient, $\frac{\partial p}{\partial z}$ $(or\frac{dp}{dz})$ is constant. Hence the equation to be tackled therefore becomes:

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\frac{dp}{dz}\cdot r \text{ , (by putting } v_z = u). \qquad \ldots(3.4)$$

Integrating equation (3.4) gives: $r\frac{du}{dr} = \frac{1}{\mu}\frac{dp}{dz}\cdot\frac{r^2}{2} + k_1$,

$$\frac{du}{dr} = \frac{1}{\mu}\frac{dp}{dz}\cdot\frac{r}{2} + \frac{k_1}{r},$$

$$u = \frac{1}{\mu}\frac{dp}{dz}\cdot\frac{r^2}{4} + k_1\ln r + k_2. \qquad \ldots(3.5)$$

The boundary conditions here are: at $r = 0$, velocity is finite; at $r = R, u = 0$.

Since u is finite it is obvious that the term containing $k_1 \ln r$ disappears, $\therefore\ k_1 = 0$.

Substituting in the second equation gives:

$$0 = \frac{1}{\mu}\frac{dp}{dz}\cdot\frac{R^2}{4} + k_2; \quad \Rightarrow k_2 = -\frac{R^2}{4\mu}\frac{dp}{dz}$$

Substituting for k_1 and k_2 gives:

$$u = \frac{r^2}{4\mu}\frac{dp}{dz} - \frac{R^2}{4\mu}\frac{dp}{dz}; \text{ or } u = -\frac{1}{4\mu}\frac{dp}{dz}\left(R^2 - r^2\right). \qquad \ldots(3.6)$$

It is obvious that from this equation the velocity distribution for the flow will be parabolic. This can be seen more clearly if the equation is expressed as:

$$u=-\frac{1}{4\mu}\frac{dp}{dz}R^2\left[1-\left(\frac{r}{R}\right)^2\right]. \qquad \ldots(3.7)$$

RESULTS

Urine and Urethra (analytical results)

What are termed solutions and sought for, are:

(i) The velocity distribution (axial velocity). This is already expressed as equation (3.7);

(ii) The maximum velocity which is expected to occur at $r = 0$, will be:

$$u_{\max}=-\frac{1}{4\mu}\frac{dp}{dz}R^2. \qquad \ldots(4.1)$$

(iii) The average velocity is obtained by summing up all the velocities over a cross section and dividing by the cross-sectional area. Thus:

$$u_{average}=\frac{\int_0^{2\pi}\int_0^R ur\,dr\,d\theta}{\int_0^{2\pi}\int_0^R r\,dr\,d\theta} = \frac{1}{8\mu}\frac{dp}{dz}R^2. \qquad \ldots(4.2)$$

(iv) The volume rate of flow (Volume Flux), Q, can then be obtained as the product of area and average velocity as obtained. Thus:

$$Q=-\frac{1}{8\mu}\frac{dp}{dz}\pi R^4. \qquad \ldots(4.4)$$

From (3.6) the pressure gradient can be obtained as:

$$-\frac{dp}{dz}=\frac{4\mu}{\left(R^2-r^2\right)}u. \qquad \ldots(4.5)$$

But then under normal circumstances and based on our intended model, the pressure gradient can be estimated by considering the difference of pressure values at the two ends of the flow tube (pipe) and divide by the total length of the tube (pipe). Thus:

$$-\frac{dp}{dz}=\frac{P_0-P_L}{L}. \qquad \ldots(4.6)$$

Hence for a physical tube (pipe) equation (3.12) can be substituted into equations (3.6) – (4.6) for correct results.

The Urethral Model

Having obtained the required relations, the next will be to apply these to the urethra, having already assumed it to be cylindrical, based on Muhammad and Sesay (2010) model. It can then be assumed that *R* will be the radius of urethra; *L*, length of the urethra; μ the kinematic viscosity; *u* the flow velocity; and P_o and P_L are the pressures at the two ends of the urethra. Thus, the following quantities can be highlighted as follows:

The urethral physiological dimensions:

These are quantities that were determined experimentally or otherwise, hence the figures can be conveniently substituted in the resultant equations to enable the evaluation of other properties.

- As stated earlier, the length of urethra shows considerable variation based on gender differences, i.e. in women, it ranges between 0.035*m* to 0.040*m*, while in males, it varies from 0.15*m* to 0.29*m* - Inman (2013). In this study therefore the maximum lengths for both genders will be considered. Thus:

$$\Rightarrow \; L_M = 0\cdot 29m \; (maximum) \; for \; male \; urethra$$
$$L_F = 0\cdot 04m \; (maximum) \; for \; female \; urethra .$$

- The male urethra is 8 – 9*mm* in diameter, while for the female it is 6*mm* in diameter – Talati J (1989). For this study therefore:

$$\Rightarrow \; R_M = 4\cdot 5\times 10^{-3} m \; (maximum) \; for \; male \; urethra$$
$$R_F = 3\cdot 0\times 10^{-3} m \; (maximum) \; for \; female \; urethra$$

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- The urine kinematic viscosity is thermo-tropic, ranging from 1.0700cSt at 20^0C (standard deviation = 0.1076); 0.8293cSt at 37^0C (standard deviation = 0.0851); and 0.6928 at 42^0 C (standard deviation = 0.0247). Considering in this case the supposed normal temperature of the human body is 37^0C, the viscosity at that temperature will selected:

  $\Rightarrow \; \mu = 8.293\times 10^{-3} cSt$. (Notice that this is more than that of water).

- It had been stated earlier that during urination the bladder empties by the muscles contracting to squeeze the urine out through the urethra, which under normal conditions, is voluntary controlled. It is this contraction that produces a force (pressure) which causes the flow of urine from the bladder

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  through the urethra to the outside of the body.

- “the average flow rate of urine for females is 15*mL/sec*, - ages 14 to 45; while for males is 21*mL/sec*. the average flow rate for females is 18*mL/sec*. – ages 45 to 65; for males is 12*mL/sec*. ages 45 to 65” – Uroflometry: MedlinePlus Medical Encyclopedia (Jan 30, 2017).

  From this data therefore, one can deduce that the volume flux, *Q* is such that:

$$\Rightarrow For \; male \; urethra: \; Q_M = 21mL/sec. \; for \; ages \; 14 \; to \; 45;$$
$$= 12mL/sec. \; for \; ages \; 46 \; to \; 65.$$
$$For \; female \; urethra \; Q_F = 15mL/sec. \; for \; ages \; 14 \; to \; 45;$$
$$= 18mL/sec. \; for \; ages \; 46 \; to \; 65.$$

Derived / estimated quantities:

It is obvious here the quantity that would be of most importance is the (generated) axial pressure gradient from the "contraction force" that initiates the emptying of the bladder (flow of urine from the bladder to outside) i.e. the urination process. It seems that all the available data listed above, the pressure gradient may be better obtained from equation (4.4):

$$Q = -\frac{1}{8\mu}\frac{dp}{dz}\pi R^4; \;\Rightarrow\; -\frac{dp}{dz}\;(i.e.\ pressure\ gradient) \;=\; \frac{8\mu}{\pi R^4}Q\,.$$

Since R (the urethral radius) and Q (the volume flux) both vary with gender and age, different results are bound to be obtained (so also as in the case of the urethral length). These can be tabulated as follows:

Table 1: Authenticity or otherwise, of our assumption on the Hagen-Poiseuille flow

	Male	**Male**	**Female**	**Female**
Age (years)	14 – 45 yrs	46 - 65	15 - 45	46 - 65
Viscosity μ (Pa.s)	8.293×10^{-3}	8.293×10^{-3}	8.293×10^{-3}	8.293×10^{-3}
Radius R (m)	4.5×10^{-3}	4.5×10^{-3}	3.0×10^{-3}	3.0×10^{-3}
Vol. Flux Q (m^3/s)	2.1×10^{-6}	1.2×10^{-6}	1.5×10^{-6}	1.8×10^{-6}
Pressure Grad. (Pascal/m)	4.8667×10^{-2}	2.710×10^{-2}	3.9107×10^{-2}	3.6210×10^{-2}

From the above table the authenticity or otherwise, of our assumption on the Hagen-Poiseuille flow can be ascertained by obtaining the corresponding values of the maximum axial velocities of urine flow, and comparing with any physically obtained figures. To test this case therefore, only one case will be considered, i.e. the case of maximum velocity condition, which will be the condition of male urethra of the 14-45 years age group. Hence using equation (4.1) and substituting the figures obtained in the table:

$$u_{\max} = -\frac{1}{4\mu}\frac{dp}{dz}R^2;$$

$$\Rightarrow u_{\max} \;=\; -\frac{1}{4}\times\frac{1}{8.293\times10^{-6}}\times 4.8667\times10^{-2}\left(4.5\times10^{-3}\right)^2 .$$

$$=\; 2\cdot9709\times10^{-2}\,ms^{-1}$$

This figure seems to agree to some extent with one experimental figure as postulated by Nurnberger N. Z Urol Nephrol, (1985), which found “the speed values for small boys to be between 235 to 325 *cm/s*”. However, one should not be oblivious of urethral diameter difference between a child and an adult.

CONCLUSION

From this study therefore, the simple results so obtained, indicate similarity with some experimental results earlier obtained in similar studies. Hence one can say that the Hagen-Poiseuille flow can be applied to the study of flow of fluids in other similar physiological situations. The study may be further extended to investigate some clinical situations that are associated with these types of flows.

REFERENCES

Batchelor, G. K. (2006) – *An Introduction to Fluid Dynamics* (Cambridge University Press).

Li, C. H (1970) – *Peristaltic Transport in Circular Cylindrical Tubes*. Journal of Biomechanics, 3, pp 513-523.

Muhammad, A. B and Sesay, M. S (2010) – *Natural Peristaltic Flows of Incompressible Newtonian Fluids: A Case For Suitable Flow Tract.* Science Forum, Journal of Pure and Applied Sciences, Vol. 13.

Spiegel, Murray S, (1968) – *Mathematical Handbook of Formulas and Tables*. P. 77; items 14.386 and 14.389. Shaum Outline Series in Mathematics – Mcgraw- Hill, New York.

Subba Reddy, M. V., Manoranjan Mishra, Sreenadh, S. and Ramachandra Rao, A. (2005) – *Influence of Lateral Walls on Peristaltic Flow in a Rectangular Duc*t. Journal of Fluid Engineering, Vol. 127, pp 824-827.

Yavorsky, B. and Detlaf, A. (1977) – *Handbook of Physics* – MIR publishers, Moscow, pp 400-401 (*19 . 5 . 3 – Axioms of Dimensional Analysis*).

Yin, F. and Fung, Y. C. (1969) – *Peristaltic Waves In Circular Cylindrical Tubes*. Journal of Applied Mechanics 36, pp 579-587.

Zien, T. F. and Ostrach, S. (1969) – *A long wave approximation to peristaltic motion* – Journal of Biomechanics, Vol. 3, PP 63-75.

Inman B.A (2013) – Impact of Temperature on urine viscosity; International Journal of Hyperthermia. 29(3).

Talati J (1989) - Journal of Pakistan Medical Association; pp 79-83.

Uroflometry: MedlinePlus Medical Encyclopedia (2017).

Nurnberger N. Z (1985): Normal values of Urine Stream Velocity in Boys. International Urology and Nephrology Journal: springer.