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Numerical Solution of Two Dimensional Mixed Volterra-Fredholm Integral Equations Using Polynomial Collocation Method

Albert A. Shalangwa[1,2,3]*, Matthew R. Odekunle[2], Solomon O. Adee[2], Cornelius Micheal[1,3] and David John[1,3]

[1]Department of Mathematical Sciences, Faculty of Science, Gombe State University, PMB 127 Tudun-Wada Gombe, Nigeria

[2]Department of Mathematics, Modibbo Adama University, Yola, Nigeria

[3]GSU-Mathematics for Innovative Research Group (GSU-MIR)

Corresponding Authors: draashalangwa2@gmail.com

ABSTRACT

In this research, polynomial collocation method was used to develop and implement the numerical solution of linear two-dimensional mixed Volterra-Fredholm integral equations (L2D MVFIE). The Integral equation was transform into systems of algebraic equations using standard collocation points with Bernstein polynomial as a basis function and then solved the linear algebraic equations using Gaussian elimination method. To illustrate the effectiveness, accuracy and efficiency of the approach, illustrative examples are provided. It demonstrates that the method converge faster to the exact solution as the value of N increases.

Keywords: Volterra Integral equation, Fredholm Integral equation, Mixed Volterra-Fredholm Integral equation, Collocation, Two-Dimensional Integral

MSC 2020 Subject Classification: Primary 45A05; Secondary 65R20

INTRODUCTION

An equation is considered integral if the unknown function appears inside the integral sign. The various forms of integral equations primarily depend on the equation's kernel and the integration's limits. According to Wazwaz (2011), an integral equation is referred to as a Volterra integral equation if at least one of the limits is variable and a Fredholm integral equation if the limits of integration are fixed. An essential tool for modeling a wide range of phenomena and resolving various boundary value problems involving ordinary and partial differential equations is the integral equation. One of the most helpful mathematical fields in both pure and applied mathematics is integral equations, which has numerous applications in science, engineering, etc. (Khuri & Wazwaz, 1996).

An equation that combines the Fredholm integral and the Volterra integral in one equation is known as the Volterra-Fredholm Integral equation. While polynomial collocation methods have been extensively studied for one-dimensional integral equations, there have been little study on the two dimensional integral equations. The majority of two-dimensional mixed Volterra-Fredholm integral equation are difficult to solve analytically, hence it is necessary to develop an accurate and efficient numerical method that will solve the problems.

Many have developed various method for solving two-dimensional mixed Volterra-Fredholm integral equations including; multiquadric radial basis functions (Almasied and Meleh, 2014), Two-dimensional Legendre wavelets method (Banifatemi et al., 2007), Applications of twodimensional triangular functions (Maleknejad and Behbahani, 2012), series solution methods (Rostam and Karzan, 2015) and many more. In this research work, we will consider Two-Dimensional linear mixed Volterra-Fredholm integral equation of the form:

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$$q(x,t) = f(x,t) + \lambda \int_0^t \int_a^b k(x,t,y,z) q(y,z) dydz \quad (1)$$

where $u(x,t)$ is considered an unknown function to be determined, the functions $f(x,t)$ is analytic on C ($[0,1]^2$, R), $k(x,t,y,z)$ is analytic on C ($[0,1]^4$, R), $u(y,z)$is a continuous function with respect to $u(y,z)$ and λ is a constant coefficient.

In order to apply the Bernstein polynomials in the interval [0, 1], $B_{i,n}(x)$ is defined as (Joy, 2000)

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, i = 0,1,2,\ldots,n \quad (2)$$

Bernstein polynomials of degree *n* in the interval [0, 1] can also be written in the following equivalent form

$$B_{i,n}(x) = \sum_{p=0}^{n-i} \binom{n}{i} \binom{n-1}{p} (-1)^p x^{i+p} \quad (3)$$

Bernstein polynomials of degree *n* can be defined recursively by blending together two Bernstein polynomials of degree $n - 1$. That is, the *kth* nth-degree Bernstein polynomial can be written as (Joy, 2000)

$$B_{k,n}(x) = (1-x)B_{k,n-1}(x) + xB_{k-1,n-1}(x), k = 0(1)n, n \geq 1 \quad (4)$$

MATERIALS AND METHODS

In this section, we developed a numerical method using polynomial collocation method for solving two Dimensional Mixed Volterra-Fredholm Integral Equations (2D MVFIE). This method is based on collocation approach and also considers linear combination of Bernstein polynomial as our approximated solution. In this section, we will develop a method by reducing the two-dimensional mixed Volterra-Fredholm integral equation to system of linear algebraic equations using standard collocation points. We recall that equation (1) is given by;

$$q(x,t) = f(x,t) + \lambda \int_0^t \int_a^b k(x,t,y,z) q(y,z) dydz \quad (5)$$

Let $U_N(x,t)$ be the approximate solution of equation (5) where

$$Q_N(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{N} c_{ij} B_{i,N}(x) B_{j,N}(t) = \phi(x,t)C \quad (6)$$

Substituting (6) into (5) we have,

$$\phi(x,t)C = f(x,t) + \lambda \int_0^t \int_a^b k(x,t,y,z)(\phi(y,z)C) dydz \quad (7)$$

$$\phi(x,t)C - \lambda \int_0^t \int_a^b k(x,t,y,z)(\phi(y,z)C) dydz = f(x,t) \quad (8)$$

$$\left(\phi(x,t) - \lambda \int_0^t \int_a^b k(x,t,y,z)\phi(y,z) dydz\right) C = f(x,t) \quad (9)$$

Collocating equation (9) using standard collocation points at $x = x_i$ and $t = t_j$ with

$$x_i = a + \frac{(b-a)i}{N}, \mathrm{i} = 0(1)\ \mathrm{N}$$

$$t_j = a + \frac{(b-a)j}{N}, \mathrm{j} = 0(1)\ \mathrm{N}$$

$$\left(\phi(x_i,t_j) - \lambda \int_0^t \int_a^b k(x_i,t_j,y,z)\phi(y,z) dydz\right) C = f(x_i,t_j) \quad (10)$$

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Where $\gamma(x_i,t_j) = \left(\phi(x_i,t_j) - \lambda\int_0^t\int_a^b k(x_i,t_j,y,z)\phi(y,z)dydz\right)$ and $C = [c_{0,0}, c_{0,1}, c_{0,2}, \ldots c_{0,N}, \ldots, c_{N,0}, c_{N,1}, c_{N,2}, \ldots, c_{N,N}]$

$$\gamma(x_i,t_j)C = f(x_i,t_j) \tag{11}$$

Multiplying both sides of equation (11) by $\gamma(x_i,t_j)^{-1}$ gives

$$C = \gamma(x_i,t_j)^{-1} f(x_i,t_j) \tag{12}$$

Substituting C into the approximate solution in equation (6) gives;

$$Q_N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}B_{i,N}(x)B_{j,N}(t) = \phi(x,t)C$$

$$= \phi(x,t)\gamma(x_i,t_j)^{-1} f(x_i,t_j); \tag{13}$$

$$i,j = 0(1)N$$

The system of equations is then solved using Maple 18 software and the unknown constants obtained are then substituted back into the approximate solution to get the required solution.

NUMERICAL EXAMPLES

In this research, numerical examples are used to test the simplicity and efficiency of the method and are presented in tables except where it gives exact solution. All computations are done with the help of MAPLE 18 software. Let $Q_N(x,t)$ and $Q(x,t)$ be the approximate and exact solution respectively then $Error_N = |Q_N(x,t) - Q(x,t)|$.

Example 1:

Rostam and Karzan (2015) considered a linear two-dimensional mixed Volterra-Fredholm integral equation of the second kind

$$q(x,t) = x^2 + xt - \frac{1}{15}xt^4 - \frac{1}{16}xt^5 + \int_0^t\int_0^1 (xty^2z^2)q(y,z)dydz \tag{14}$$

which has an exact solution given as $q(x,t) = x^2 + xt$ in the interval $(x,t) = [0,1]$.

Case1: Using Bernstein Polynomial Basis Function

Let the approximate solution of equation (14) be;

$$Q_1(x,t) = \sum_{i=0}^{1}\sum_{j=0}^{1} c_{i,j}B_{i,1}(x)B_{j,1}(t) \tag{15}$$

Substituting equation (15) in equation (14) we have;

$$\sum_{i=0}^{1}\sum_{j=0}^{1} c_{i,j}B_{i,1}(x)B_{j,1}(t)$$

$$= x^2 + xt - \frac{1}{15}xt^4 - \frac{1}{16}xt^5$$

$$+ \int_0^t\int_0^1 (xty^2z^2)\left(\sum_{i=0}^{1}\sum_{j=0}^{1} c_{i,j}B_{i,1}(y)B_{j,1}(z)\right)dydz \tag{16}$$

Collocating equation (16) using standard collocation points at $x = x_i$ and $t = t_i$ and solving using maple 18 software gives;

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$$Q_1(x,t)=\frac{229}{225}xt+x$$

Let the approximate solution of equation (14) for $N=3$ be

$$Q_3(x,t)=\sum_{i=0}^{3}\sum_{j=0}^{3}c_{i,j}B_{i,3}(x)B_{j,3}(t) \quad (17)$$

Substituting equation (17) in equation (14) we have;

$$\sum_{i=0}^{3}\sum_{j=0}^{3}c_{i,j}B_{i,3}(x)B_{j,3}(t) = x^2+xt-\frac{1}{15}xt^4-\frac{1}{16}xt^5 + \int_0^t\int_0^1 (xty^2z^2)\left(\sum_{i=0}^{3}\sum_{j=0}^{3}c_{i,j}B_{i,3}(y)B_{j,3}(z)\right)dydz \quad (18)$$

Collocating equation (18) using standard collocation points at $x=x_i$ and $t=t_i$ and solving using maple 18 software gives;

$$Q_3(x.t)=\frac{-221}{720}t^3x+\frac{390014479501}{1229996259000}tx^3+\frac{119}{540}t^2x-\frac{67201657133}{295199102160}tx+x^2$$

Let the approximate solution of equation (14) for $N=5$ be;

$$Q_5(x,t)=\sum_{i=0}^{5}\sum_{j=0}^{5}c_{i,j}B_{i,5}(x)B_{j,5}(t) \quad (19)$$

Substituting equation (19) in equation (14) we have;

$$\sum_{i=0}^{5}\sum_{j=0}^{5}c_{i,j}B_{i,5}(x)B_{j,5}(t) = x^2+xt-\frac{1}{15}xt^4-\frac{1}{16}xt^5 + \int_0^t\int_0^1 (xty^2z^2)\left(\sum_{i=0}^{5}\sum_{j=0}^{5}c_{i,j}B_{i,5}(y)B_{j,5}(z)\right)dydz \quad (20)$$

Collocating equation (20) using standard collocation points at $x=x_i$ and $t=t_i$ and solving using maple 18 software gives;

$$Q_5(x.t)=-\frac{1}{16}t^5x-\frac{1}{15}t^4x-\frac{36737}{800000}tx^3+\frac{518429}{42000000}tx^2+\frac{499429}{500000}tx+x^2+\frac{166529}{1200000}tx^4+\frac{81872321}{3255000000}tx^5$$

Case 2: Using Power Series Basis Function

Let the approximate solution of equation (14) for $N = 1,3\ and\ 5$ be;

$$Q_1(x,t)=\sum_{i=0}^{1}\sum_{j=0}^{1}c_{i,j}x^it^j \quad (21)$$

$$Q_3(x,t)=\sum_{i=0}^{3}\sum_{j=0}^{3}c_{i,j}x^it^j \quad (22)$$

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$$Q_5(x,t)=\sum_{i=0}^{5}\sum_{j=0}^{5}c_{i,j}x^it^j \quad (23)$$

Substituting equation (21), (22) and (23) in equation (14) we have;

$$\sum_{i=0}^{1}\sum_{j=0}^{1}c_{i,j}x^it^j=x^2+xt-\frac{1}{15}xt^4-\frac{1}{16}xt^5 + \int_0^t\int_0^1 (xty^2z^2)\left(\sum_{i=0}^{1}\sum_{j=0}^{1}c_{i,j}x^it^j\right)dydz \quad (24)$$

$$\sum_{i=0}^{3}\sum_{j=0}^{3}c_{i,j}x^it^j=x^2+xt-\frac{1}{15}xt^4-\frac{1}{16}xt^5 + \int_0^t\int_0^1 (xty^2z^2)\left(\sum_{i=0}^{3}\sum_{j=0}^{3}c_{i,j}x^it^j\right)dydz \quad (25)$$

$$\sum_{i=0}^{5}\sum_{j=0}^{5}c_{i,j}x^it^j=x^2+xt-\frac{1}{15}xt^4-\frac{1}{16}xt^5 + \int_0^t\int_0^1 (xty^2z^2)\left(\sum_{i=0}^{5}\sum_{j=0}^{5}c_{i,j}x^it^j\right)dydz \quad (26)$$

Collocating equation (24), (25) and (26) using standard collocation points at $x=x_i$ and $t=t_i$ and solving using maple 18 software gives;

$$Q_1(x,t)=\frac{2771}{2775}xt+\frac{1}{3}x$$

$$Q_3(x,t)=\frac{33794573}{33858000}xt+\frac{119}{540}xt^2-\frac{221}{720}xt^3+x^2-\frac{1728101}{8125920}x^2t+\frac{815677}{2708640}x^3t$$

$$Q_5(x,t)=\frac{166529}{1200000}x^4t+\frac{81872321}{3255000000}x^5t-\frac{36737}{800000}x^3t+\frac{518429}{42000000}x^2t+\frac{499429}{500000}xt + x^2-\frac{1}{16}t^5x-\frac{1}{15}t^4x$$

Table 1: Results of Example 1 using Bernstein Polynomial

(x, t)	Exact	N = 1	N = 3	N = 5
(0,0)	0.0000000000	0.0000000000	0.0000000000	0.0000000000
(0.1,0.1)	0.0200000000	0.1101777778	0.02000706462	0.01999701517
(0.2,0.2)	0.0800000000	0.2407111111	0.08001131539	0.08000027845
(0.3,0.3)	0.1800000000	0.3916000000	0.1800055923	0.1800065287
(0.4,0.4)	0.3200000000	0.5628444444	0.3200070749	0.3200170898
(0.5,0.5)	0.5000000000	0.7544444444	0.5000572821	0.5000571736
(0.6,0.6)	0.7200000000	0.9664000000	0.7202220726	0.7201682990
(0.7,0.7)	0.9800000000	1.198711111	0.9805916442	0.98037381870
(0.8,0.8)	1.2800000000	1.451377778	1.281280534	1.280617502000
(0.9,0.9)	1.6200000000	1.724400000	1.622427621	1.620675350000
(1.0,1.0)	2.0000000000	2.017777778	2.004196119	2.000040620000

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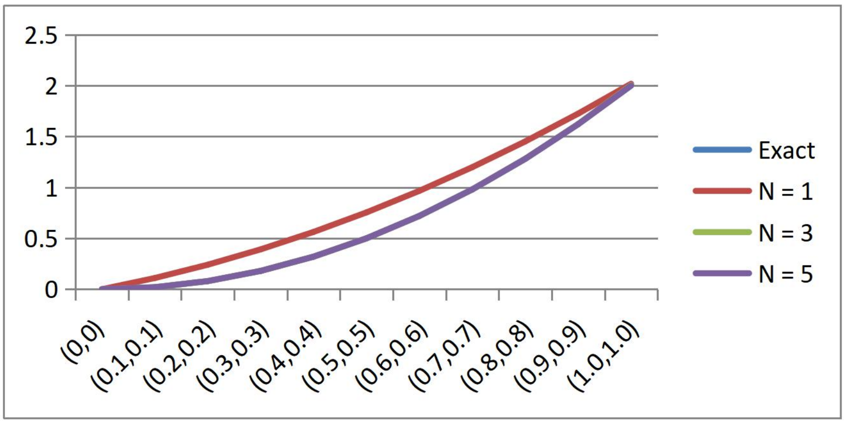

Figure 1: Graph of Results of Example 1 using Bernstein Polynomials

Table 2: Results of Example 1 using Power Series

(x, t)	Exact	N = 1	N = 3	N = 5
(0,0)	0.0000000000	0.0000000000	0.0000000000	0.0000000000
(0.1,0.1)	0.0200000000	0.04331891892	0.01998839131	0.01999701515
(0.2,0.2)	0.0800000000	0.1066090090	0.07997741902	0.08000027856
(0.3,0.3)	0.1800000000	0.1898702703	0.1799924139	0.1800065288
(0.4,0.4)	0.3200000000	0.2931027027	0.3200447737	0.3200170896
(0.5,0.5)	0.5000000000	0.4163063063	0.5001319626	0.5000571745
(0.6,0.6)	0.7200000000	0.5594810811	0.7202375120	0.7201682999
(0.7,0.7)	0.9800000000	0.7226270270	0.9803310198	0.9803738099
(0.8,0.8)	1.2800000000	0.9057441442	1.280368151	1.2806175100
(0.9,0.9)	1.6200000000	1.108832432	1.620290638	1.6206754120
(1.0,1.0)	2.0000000000	1.331891892	2.000026276	2.000040584

Table 3: Absolute Error of Example 1 with N=5

(x, t)	Exact	$Error_B$	$Error_P$
(0,0)	0.0000000000	0.0000000000	0.0000000000
(0.1,0.1)	0.0200000000	2.98483e-6	2.98485e-6
(0.2,0.2)	0.0800000000	2.7845e-7	2.7856e-7
(0.3,0.3)	0.1800000000	6.5287e-6	6.5288e-6
(0.4,0.4)	0.3200000000	1.70898e-5	1.70896e-5
(0.5,0.5)	0.5000000000	5.71736e-5	5.71745e-5
(0.6,0.6)	0.7200000000	1.68299e-4	1.682999e-4
(0.7,0.7)	0.9800000000	3.738187e-4	3.738099e-4
(0.8,0.8)	1.2800000000	6.17502e-4	6.17510e-4
(0.9,0.9)	1.6200000000	6.7535e-4	6.75412e-4
(1.0,1.0)	2.0000000000	4.062e-5	4.0584e-5

Example 2:

Narges et al. (2019) considered a linear two-dimensional mixed VolterraFredholm integral equation of the second kind;

$$q(x,t) = x^2 + xt - (1.15)xt^4 - (1.16)xt^5 + \int_0^t \int_0^1 xty^2z^2q(y,z)dydz \quad (27)$$

which has an exact solution given as $u(x,t) = x^2 + xt$ in the interval $x, t = [0, 1]$. Let the approximate solution of equation (14) be;

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$$Q_5(x,t) = \sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j}B_{i,1}(x)B_{j,1}(t) \quad (28)$$

Substituting equation (28) in equation (28) gives;

$$\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j}B_{i,1}(x)B_{j,1}(t)$$
$$= x^2 + xt - (1.15)xt^4 - (1.16)xt^5 + \int_0^t\int_0^1 xty^2z^2 \sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j}B_{i,1}(y)B_{j,1}(z)\,dydz \quad (27)$$

Collocating equation (27) using standard collocation points at $x = x_i$ and $t = t_i$ and solving using maple 18 software gives;

$$Q_5(x.t) =- 1.15000000\mathrm{t}^4\mathrm{x} - 1.160000000\mathrm{t}^5\mathrm{x} + 1.0 \times 10^{-8}\mathrm{t}^4\mathrm{x}^2 - 1.82 \times 10^{-8}\mathrm{t}^5\mathrm{x}^5 + 2.0 \times 10^{-8}t^5x^4 + 1.00 \times 10^{-7}t^4x^3 + 1.00 \times 10^{-7}t^4x^4 + 1.00 \times 10^{-7}t^3x^5 - 1.00 \times 10^{-7}t^4x^3 + 2.00 \times 10^{-8}t^2x^5 + 1.2 \times 10^{-7}t^2x^4 - 0.682951830tx^5 - 2.00 \times 10^{-8}t^3x^2 - 6.00 \times 10^{-8}t^2x^3 + 1.36107066tx^4 + 5.505714155 \times 10^{-15}t^3x + 1.00 \times 10^{-8}t^2x^2 - 0.82484079tx^3 - 5.505714155 \times 10^{-15}t^2x + 0.221798170tx^2 + 0.9794752000tx - 5.505714155 \times 10^{-16}x + 1.000000000x^2$$

Table 4: Results of Example 2 using Bernstein Polynomial

(x,t)	Exact	Approximate (N=5)	Error (N=5)
(0,0)	0.000000000	0.0000000000	0.0000000000
(0.1,0.1)	0.020000000	0.01993433385	0.00006566615
(0,0.3)	0.09000000000	0.09000000000	0.00000000000
(0.1,0.3)	0.12000000000	0.1200518993	0.00005189930
(0,0.5)	0.25000000000	0.2500000000	0.00000000000
(0.1,0.5)	0.30000000000	0.3005173716	0.0005173716
(0,0.7)	0.49000000000	0.4900000000	0.0000000000
(0.1,0.7)	0.56000000000	0.5622516505	0.0022516505
(0,0.9)	0.81000000000	0.8100000000	0.0000000000
(0.1,0.9)	0.90000000000	0.9048458102	0.0048458102

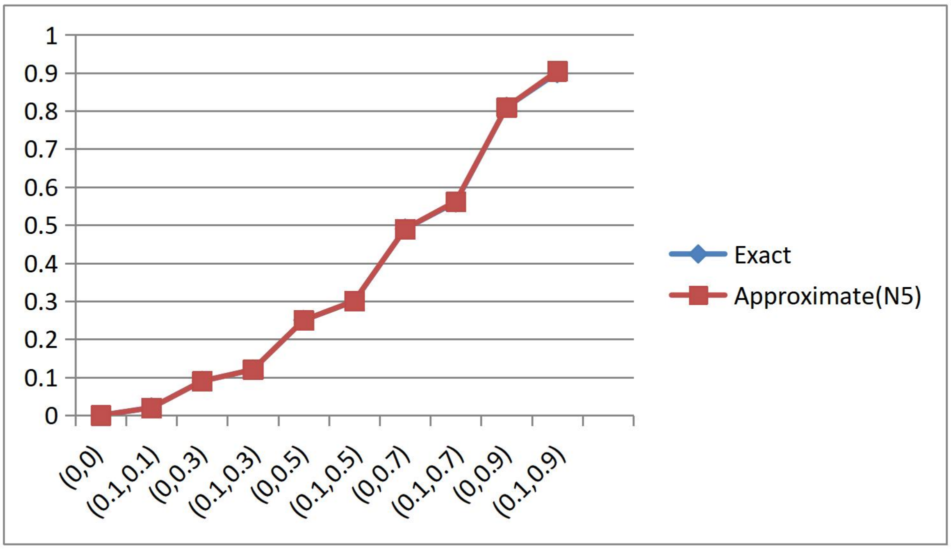

Figure 2: Results of Example 2 using Bernstein Polynomial with N=5

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RESULTS AND DISCUSSION

It was noticed from the results obtained for example 1 as shown in Table 1 and figure 1, it shows that as N increases from $N = 1$, $N = 3$ and $N = 5$ the result converges faster to the exact solution in respective of whether power series was used or Bernstein Polynomial. For instance, for $(x, t) = (1.0, 1.0)$ and $N = 1$, $N = 3$, $N = 5$ with Bernstein Polynomial, the result obtained is given as 2.017777778, 2.004196119, and 2.000040620000 respectively. Similarly, for $(x, t) = (1.0, 1.0)$ and $N = 1$, $N = 3$, $N = 5$ with Power Series, the result obtained is given as 1.331891892, 2.000026276, and 2.000040584 respectively, this shows that the best results were obtained when $N = 5$ when compared with the results for $N = 1$ and $N = 3$ which implies the convergence, efficiency and reliability of the method developed.

In example 2, the absolute errors show that at $N = 5$ and at different values of (x, t) from Table 2 and figure 2 shows that the method is consistent, stable and converge to the exact solution.

CONCLUSION

A Numerical method is developed for solving two dimensional mixed Volterra-Fredholm integral equations using polynomial collocation method, MAPLE codes was written to implement the developed method. The numerical method developed is reliable, efficient and easy to compute. The results of the numerical examples solved shows that the method is suitable for the numerical examples solved.

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