Bima Journal of Science and Technology, Vol. 3(1) July, 2019. ISSN: 2536-6041

PERISTALTIC FLOW OF PARTICLE-FLUID SUSPENSION IN A CIRCULAR CYLINDRICAL TRACT

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ABSTRACT

Peristalsis is a form of fluid transport achieved with the aid of progressive, contraction-expansion wavelike movements along the walls of the fluid containing tract (channel/pipe/tube). It appears mostly in biological systems, such as in the ureter, in the intestines and oviducts in the human body. In the study, a particle-fluid suspension is treated as a two-phase fluid and considered to be flowing through a circular cylindrical flow tract, under the influence of a long wave peristaltic flow situation. Two sets of equations, for the fluid phase and the particulate phase, are taken. At the end when the volume fraction of the particulate phase q is made zero, the results did agree, to some extent with the case of a single-phase fluid. From this study therefore it implies that, as long as a Newtonian fluid is assumed, the fluid type may not adversely affect the results, hence may correspond to the single-phase fluid situation.

Keywords: Contraction, Long wave, Fluid transport, Flow situation, Peristaltic

INTRODUCTION

Peristalsis is a form of fluid flow/transport mechanism achieved with the aid of a progressive contraction – expansion wavelike situation along the walls of the fluid-containing tract, in this case, cylindrical (tube-like/pipe). This mechanism is known to be one of the major fluid transport processes in many biological systems. It occurs as involuntary movements of the longitudinal and circular muscles, primarily in the digestive tract, but occasionally in other hollow tubes of the body, that occur in progressive wavelike contractions. The waves can be short local reflexes, or long continuous contractions that travel the whole length of the organ, depending upon their location and what initiates their action. In particular, peristaltic transport mechanism is involved in urine transport from kidney to bladder through the ureter and movement of chime in the gastro-intestinal tract. The peristalsis phenomenon can be represented diagrammatically as figure 1.

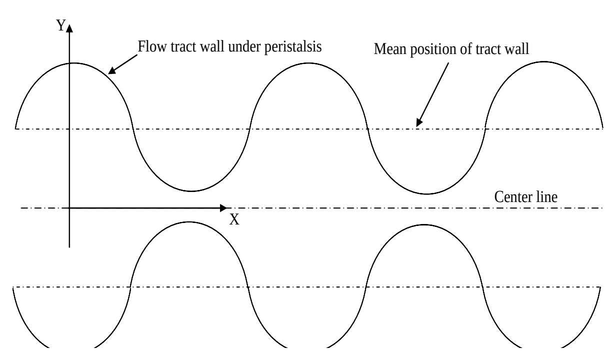

Figure 1: Peristaltic flow process – 2-dimensional flow

Li (1970), adopted a long wave approximation, based on the Zien and Ostrach, (1969) model, to analyze the peristaltic pumping of a Newtonian Fluid, in a circular cylindrical tube. Essentially, it is the application of the Zien and Ostrach, (1969) analysis to the axisymmetric case. The same assumptions that the length of the peristaltic wave is large compared with mean radius of the tube; and the frequency of the peristaltic wave is small as compared to the reciprocal of the characteristic time for the vorticity diffusion – thus making the Reynolds number small; are again adopted. The perturbation solutions techniques are also employed, with closed-form solutions, up to the second order, obtained; and criteria for backward flow discussed.

Other important related studies on this topic of recent years include Srivastava and Srivastava, (1997); Mekheimer, (1998); Srivastava, (2002); Srivastava, (2007); and Medhavi et al. (2009). It is noteworthy that all these used a circular cylindrical model, while Muhammad, (2010), used the rectangular (channel) model. In this study however, only the zeroth order solutions will be presented as these “are more applicable to physical problems”- Muhammad and Sesay, (2010). This study aimed at analyzing the peristaltic transport of a particle-fluid suspension in the same methods as carried out by Li (1970) for a circular cylindrical tube. Whereas in the previous study, a single phase, incompressible Newtonian fluid was used, in this case, a particle-fluid suspension, considered as a Newtonian fluid, is used. The suspension is considered as a two-phase fluid, that is, the fluid phase and the particle phase. The same calculation processes will be followed, in each case, with the hope of getting parallel results as previously obtained.

MATERIALS AND METHOD

Problem Formulation

A viscous incompressible Newtonian fluid, consisting of a fluid matter in which uniform rigid spherical particles are suspended, to form a two-phase fluid, is considered. The fluid is assumed to flow through a circular cylindrical tract (in form of a tube or pipe), with flexible walls, upon which symmetric, traveling transverse waves are imposed. The two dimensional set up of the model, is shown in figure 2.

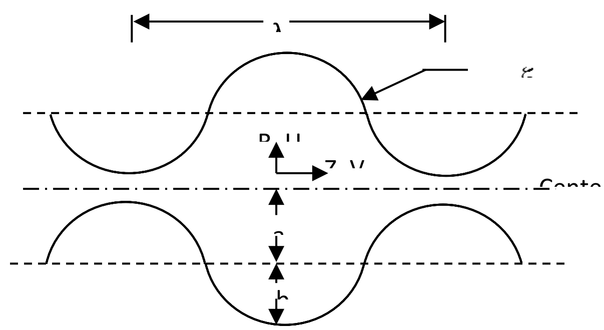

Figure 2: A Two-Dimensional Tube Under Peristalsis

Definition of Parameters

(i) Since a two-dimensional axi-symmetrical circular cylindrical tube/pipe is being considered, a cylindrical coordinate system (R, Z) is chosen; with the Z-axis in the direction of wave propagation, hence aligned with the centre line of the tract. The

R-axis is in the radial direction, normal to the mean position of the tract walls.

(ii) U and V are the velocity components of the fluid phase in the R and Z directions respectively; while U_p and V_p for the particulate phase.

(iii) The actual density of the material constituting the fluid phase is denoted by ρ_f, while that of the particle phase is denoted by ρ_p. If the volume fraction of the particle phase is denoted by q, then the volume fraction of the fluid phase will be $(1-q)$. Hence the corresponding effective densities will be $q\rho_p$ for the particle phase; and $(1-q)\rho_f$ for the fluid phase.

(iv) The effective viscosity of the suspension is denoted by $\mu_s(q)$. For this problem, an empirical relation for the viscosity of the suspension is:

$$\mu_s(q) = \frac{\mu_0}{1-nq}; \text{ Such that: } n = 0\cdot 070\exp\left[2\cdot 49q + \frac{1107}{T}exp\left(-1\cdot 69q\right)\right].$$

Where T is absolute temperature$\left({}^0K\right)$, as presented by Charm and Kurland (1974).

The expected drag coefficient will be $F_0 = \frac{9}{2}\frac{\mu_0}{l^2}\lambda'(q)$; such that:

$$\lambda'(q) = \frac{4+3\left(8q-3q^2\right)^{\frac{1}{2}}+3q}{\left(3-3q\right)^2};$$ μ_0 is the fluid viscosity; l the radius of a particle.

This relation represents the classical Stoke's drag for small particle Reynolds number, modified to account for the finite particulate fractional volume, through the function $\lambda'(q)$, as obtained by Tam (1969).

Equations of Motion

The peristaltic motion of the tract walls can be represented by:

$$R = \xi\left(Z,T\right) = \pm\left[a + b\sin\frac{2\pi}{\lambda}\left(Z-cT\right)\right]$$

; such that a is the radius of the cylindrical tract (pipe or tube), b is the amplitude, λ is the wavelength, c the wave speed.

For the motion of the suspension, two sets of equations, for the fluid phase and the particulate phase, are considered. The equations of motion, based on Drew (1979); Srivastava and Srivastava (1984), are as follows:

(a) *For the Fluid phase*:

$$(1-q)\rho_f\left(\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial R}+V\frac{\partial U}{\partial Z}\right)$$
$$= -(1-q)\frac{\partial P}{\partial R}+\mu_S(1-q)\left(\frac{\partial^2 U}{\partial R^2}+\frac{1}{R}\frac{\partial U}{\partial R}-\frac{U}{R^2}+\frac{\partial^2 U}{\partial Z^2}\right)+ qF_0\left(U_p-U\right) \quad ...(1)$$

$$(1-q)\rho_f\left(\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial R}+V\frac{\partial V}{\partial Z}\right)$$
$$= -(1-q)\frac{\partial P}{\partial Z}+\mu_S(1-q)\left(\frac{\partial^2 V}{\partial R^2}+\frac{1}{R}\frac{\partial V}{\partial R}-\frac{V}{R^2}+\frac{\partial^2 V}{\partial Z^2}\right)+ qF_0\left(V_p-V\right) \quad ...(2)$$

Continuity Equation: $(1-q)\left(\frac{\partial U}{\partial R}+\frac{U}{R}+\frac{\partial V}{\partial Z}\right) = 0.$ ...(3)

(b) *For the Particle phase*:

$$q\rho_p\left(\frac{\partial U_p}{\partial T}+U_p\frac{\partial U_p}{\partial R}+V_p\frac{\partial U_p}{\partial Z}\right) = -q\frac{\partial P}{\partial R}+ qF_0\left(U-U_p\right) \quad ...(4)$$

$$q\rho_p\left(\frac{\partial V_p}{\partial T}+U_p\frac{\partial V_p}{\partial R}+V_p\frac{\partial V_p}{\partial Z}\right) = -q\frac{\partial P}{\partial Z}+ qF_0\left(V-V_p\right) \quad ...(5)$$

Continuity Equation: $q\left(\frac{\partial U_p}{\partial R}+\frac{U_p}{R}+\frac{\partial V_p}{\partial Z}\right) = 0.$ ...(6)

The boundary conditions that must be satisfied by the fluid phase on the walls are the no-slip and impermeability conditions. These can be represented by:

$U=0$ on $R=\xi(Z,T)$; and $V=\pm\frac{2\pi ca}{\lambda}\sin\frac{2\pi}{\lambda}(Z-cT)$ on $R=\xi(Z,T)$. ...(7)

However, for the particle phase, the no-slip condition may not apply. Thus, even if the cylinder wall is considered to have transverse displacement only, U_p is not necessary zero on the walls. Hence the boundary conditions on the wall will be :

$V=V_p=\pm\frac{2\pi ca}{\lambda}\sin\frac{2\pi}{\lambda}(Z-cT)$ on $R=\eta(Z,T)$. ...(8)

Since the flow is steady and axi-symmetric, we can introduce the stream function, Ψ, from Schlichting (1960), such that: $U=\frac{1}{R}\frac{\partial\Psi}{\partial Z}$, $U_p=\frac{1}{R}\frac{\partial\Psi_p}{\partial Z}$; and $V=-\frac{1}{R}\frac{\partial\Psi}{\partial R}$, $V_p=-\frac{1}{R}\frac{\partial\Psi_p}{\partial R}$.

Substituting these into the two sets of respective equations and eliminating the pressure terms gives the following equations:

(a) *For the Fluid phase*:

$$(1-q)\rho_f\left[\bar{\nabla}^2\frac{\partial\Psi}{\partial T}+\frac{1}{R}\frac{\partial\Psi}{\partial Z}\left(\bar{\nabla}^2\frac{\partial\Psi}{\partial R}-\frac{2}{R}\bar{\nabla}^2\Psi+\frac{1}{R^2}\frac{\partial\Psi}{\partial R}\right)-\frac{1}{R}\frac{\partial\Psi}{\partial R}\bar{\nabla}^2\frac{\partial\Psi}{\partial Z}\right]$$
$$=\mu_s\bar{\nabla}^2\bar{\nabla}^2\Psi+qF_0\bar{\nabla}^2\left(\Psi_p-\Psi\right). \quad ...(9)$$

(b) *For the Particle phase*:

$$q\rho_p\left[\bar{\nabla}^2\frac{\partial\Psi_p}{\partial T}+\frac{1}{R}\frac{\partial\Psi_p}{\partial Z}\left(\bar{\nabla}^2\frac{\partial\Psi_p}{\partial R}-\frac{2}{R}\bar{\nabla}^2\Psi_p+\frac{1}{R^2}\frac{\partial\Psi_p}{\partial R}\right)-\frac{1}{R}\frac{\partial\Psi_p}{\partial R}\bar{\nabla}^2\frac{\partial\Psi_p}{\partial Z}\right]$$
$$=qF_0\bar{\nabla}^2\left(\Psi-\Psi_p\right). \quad ...(10)$$

Where $\bar{\nabla}^2 \equiv \frac{\partial^2}{\partial Z^2}+\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}$.

The boundary conditions on the walls in terms of the stream function will be:

$$\left.\begin{aligned}&\frac{\partial\Psi}{\partial R}=0\\&\frac{\partial\Psi}{\partial Z}=\frac{\partial\Psi_p}{\partial Z}=\frac{2\pi}{\lambda}ac\sin\frac{2\pi}{\lambda}\left(Z-cT\right)\end{aligned}\right\} \quad ...(11)$$

The expression for the axial pressure gradients can be obtained as follows:

(i) *For the fluid phase*;

We take the axial flow i.e. equation (2), which is expressed as:

$$\left(1-q\right)\frac{1}{\rho_f}\frac{\partial P}{\partial Z}=-(1-q)\left(\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial R}+V\frac{\partial V}{\partial Z}\right)$$
$$+(1-q)\nu_s\left(\frac{\partial^2 V}{\partial R^2}+\frac{1}{R}\frac{\partial V}{\partial R}-\frac{V}{R^2}+\frac{\partial^2 V}{\partial Z^2}\right)+\frac{qF_0}{\rho_f}\left(V_p-V\right) \quad ...(12)$$

Introducing the stream function in equation (12) gives:

$$\left(1-q\right)\frac{1}{\rho_f}\frac{\partial P}{\partial Z}=-(1-q)\frac{1}{R}\left(-\frac{\partial^2\Psi}{\partial T\partial R}+\frac{1}{R^2}\frac{\partial\Psi}{\partial Z}\frac{\partial\Psi}{\partial R}-\frac{1}{R}\frac{\partial\Psi}{\partial Z}\frac{\partial^2\Psi}{\partial R^2}+\frac{1}{R}\frac{\partial\Psi}{\partial R}\frac{\partial^2\Psi}{\partial Z\partial R}\right)$$
$$+(1-q)\nu_s\frac{1}{R}\left(-\frac{1}{R^2}\frac{\partial\Psi}{\partial R}+\frac{1}{R}\frac{\partial^2\Psi}{\partial R^2}-\frac{\partial^3\Psi}{\partial R^3}+\frac{\partial^3\Psi}{\partial Z^2\partial R}\right)$$
$$+\frac{qF_0}{\rho_f}\frac{1}{R}\left(\frac{\partial\Psi_p}{\partial R}-\frac{\partial\Psi}{\partial R}\right) \quad ...(13)$$

(ii) *For the Particle phase*;

Again, we take the axial flow case, i.e. equation (5), which is expressed as:

$$q\frac{1}{\rho_p}\frac{\partial P}{\partial Z}=-q\left(\frac{\partial V_p}{\partial T}+U_p\frac{\partial V_p}{\partial R}+V_p\frac{\partial V_p}{\partial Z}\right)+q\frac{F_0}{\rho_p}\left(V-V_p\right) \quad ...(14)$$

Introducing the stream function into equation (14) gives;

$$q\frac{1}{\rho_p}\frac{\partial P}{\partial Z}=-q\frac{1}{R}\left(-\frac{\partial^2\Psi_p}{\partial T\partial R}+\frac{1}{R^2}\frac{\partial\Psi_p}{\partial Z}\frac{\partial\Psi_p}{\partial R}-\frac{1}{R}\frac{\partial\Psi_p}{\partial Z}\frac{\partial^2\Psi}{\partial R^2}+\frac{1}{R}\frac{\partial\Psi_p}{\partial R}\frac{\partial^2\Psi_p}{\partial Z\partial R}\right)$$
$$-q\frac{F_0}{\rho_p}\frac{1}{R}\left(\frac{\partial\Psi}{\partial R}-\frac{\partial\Psi_p}{\partial R}\right) \quad ...(15)$$

Non-Dimensional Formulations

Another process of modification of the resultant equations as obtained is the introduction of non-dimensional variables. This is a common technique used (mostly in Physics and Mathematics) to formulate quantities (or variables as in this case) whose magnitudes will be independent of the system of units being used, based on the following axioms; Yavorsky and Detlaf, (1980):

"The numerical value of a physical quantity A is equal to the ratio of this quantity to its unit of measurement [A]. Thus $a=\frac{A}{[A]}$.

A physical quantity does not depend upon the choice of its unit of measurement. This implies that when the unit of measurement is increased q times, the numerical value of the quantity is reduced to $\frac{1}{q}$ times of its former value.

A mathematical description of some physical phenomenon shows that the functional relationship between the numerical values of physical quantities is independent of the choice of the units of measurement of these quantities. Consequently, all the terms of an equation that describe a physical process should have the same dimensions, since this enables them to be converted to the dimensionless form by dividing both sides of the equation by some constants having the same dimensions."

The non-dimensional or dimensionless variables are obtained by dividing each variable by a 'characteristic' variable on the same scale to eliminate the dimensions of such variable. Thus the quantities become:

$r=\frac{R}{a};\ z=\frac{Z}{\lambda};\ t=\frac{c}{\lambda};\ u=\frac{U}{c};\ v=\frac{V}{c}$. Hence, we have $\psi=\frac{\Psi}{a^2c}$; and the wave function becomes $\eta(z,t)=\frac{\xi(Z,T)}{a}$.

This results in the emergence of new parameters:

$$\in=\frac{b}{a} \text{ (the amplitude ratio); } \delta=\frac{a}{\lambda} \text{ (the wave number); and}$$

$$Re=(1-q)\frac{a\rho_s c}{\mu_s}=(1-q)\frac{ac}{\upsilon} \text{ (the suspension Reynolds number)}$$

Substituting these dimensionless parameters into equation (9) gives:

$$(1-q)\left[\nabla^2\frac{\partial\psi}{\partial t}+\frac{1}{r}\frac{\partial\psi}{\partial z}\left(\nabla^2\frac{\partial\psi}{\partial r}-\frac{2}{r}\nabla^2\psi+\frac{1}{r^2}\frac{\partial\psi}{\partial r}\right)-\frac{1}{r}\frac{\partial\psi}{\partial r}\nabla^2\frac{\partial\psi}{\partial z}\right]$$

$$=\frac{1}{\delta}\frac{1}{Re}\nabla^2\nabla^2\psi+qk_f\nabla^2\left(\psi_p-\psi\right). \quad ...(16)$$

For the particle phase, equation (10) becomes:

$$q\left[\nabla^2\frac{\partial\psi_p}{\partial t}+\frac{1}{r}\frac{\partial\psi_p}{\partial z}\left(\nabla^2\frac{\partial\psi_p}{\partial r}-\frac{2}{r}\nabla^2\psi_p+\frac{1}{r^2}\frac{\partial\psi_p}{\partial r}\right)-\frac{1}{r}\frac{\partial\psi_p}{\partial r}\nabla^2\frac{\partial\psi_p}{\partial z}\right]$$

$$=qk_p\nabla^2\left(\psi-\psi_p\right). \quad ...(17)$$

Where;

$$k_f=\frac{\lambda F_0}{c\rho_f};\ k_p=\frac{\lambda F_0}{c\rho_p};\quad \nabla^2\equiv\delta^2\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r} \text{ or } \delta^2\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial r^2}+r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right).$$

For the axial pressure gradient in dimensionless parameters;

From equation (13) we obtain pressure gradient for the fluid phase as:

$$(1-q)\frac{a}{c^2\rho_f}\frac{\partial P}{\partial z}=-(1-q)\delta\frac{1}{r}\left(\frac{\partial^2\psi}{\partial t\partial r}-\frac{1}{r^2}\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial r}+\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial z\partial r}\right)$$

$$+\frac{1}{Re}\frac{1}{r}\left(-\frac{1}{r^2}\frac{\partial\psi}{\partial r}+\frac{1}{r}\frac{\partial^2\psi}{\partial r^2}-\frac{\partial^3\psi}{\partial r^3}-\delta^2\frac{\partial^3\psi}{\partial z^2\partial r}\right)$$

$$+q\chi_f\frac{1}{r}\left(\frac{\partial\psi_p}{\partial r}-\frac{\partial\psi}{\partial r}\right). \quad ...(18)$$

From equation (15) we obtain for the particle phase:

$$q\frac{a}{c^2\rho_p}\frac{\partial P}{\partial z}=q\frac{1}{r}\left(\frac{\partial^2\psi_p}{\partial t\partial r}-\frac{1}{r^2}\frac{\partial\psi_p}{\partial z}\frac{\partial\psi_p}{\partial r}+\frac{1}{r}\frac{\partial\psi_p}{\partial z}\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_p}{\partial r}\frac{\partial^2\psi_p}{\partial z\partial r}\right)$$

$$-\ q\chi_p\frac{1}{r}\left(\frac{\partial\psi}{\partial r}-\frac{\partial\psi_p}{\partial r}\right). \quad ...(19)$$

Where: $\chi_f=\frac{aF_0}{c\rho_f};\quad \chi_p=\frac{aF_0}{c\rho_p}$

The boundary conditions become:

$$\left.\begin{cases}\dfrac{\partial\psi}{\partial r}=0\\ \dfrac{\partial\psi}{\partial z}=\dfrac{\partial\psi_p}{\partial z}=2\pi r\sin 2\pi(z-t)\end{cases}\right\}at\ r=1+\varepsilon\cos 2\pi(z-t);\quad 0<\varepsilon<1 \quad ...(20)$$

Method of Solution

Following Li's (1970) method, a long wave approximation, where δ << 1, is made. Thus solutions for the stream functions ψ and ψ_p are sought for in the power series of δ. Such that:

$$\psi=\psi_0+\delta\psi_1+\delta^2\psi_2+\cdots \quad ...(21)$$

$$\psi_p=\psi_{p0}+\delta\psi_{p1}+\delta^2\psi_{p2}+\cdots \quad ...(22)$$

Substituting equations (21) and (22) into the dimensionless equations previously obtained respectively; collecting and equating coefficients of equal powers of δ on both sides of the respective equations; for up to δ^2; sequences of equations are obtained as follows:

(a) *For the fluid phase*:

Substituting equation (21) into equation (16) gives:

$$\frac{1}{Re}L\psi_0 + q\chi_f D\left(\psi_{p0} - \psi_0\right) = 0. \qquad ...(23)$$

$$\frac{1}{Re}L\psi_0 + q\chi_f D\left(\psi_{p1} - \psi_1\right)$$
$$= (1-q)\left[D\frac{\partial\psi_0}{\partial t} + \frac{1}{r}\frac{\partial\psi_0}{\partial z}\left(\frac{\partial^3\psi_0}{\partial r^3} - \frac{3}{r}\frac{\partial^2\psi_0}{\partial r^2} - \frac{3}{r^2}\frac{\partial\psi_0}{\partial r}\right) - \frac{1}{r}\frac{\partial\psi_0}{\partial r}D\frac{\partial\psi_0}{\partial z}\right]. \qquad ...(24)$$

$$\frac{1}{Re}\left(L\psi_2 + 2D\frac{\partial^2\psi_0}{\partial z^2}\right) + q\chi_f\left[\frac{\partial^2\left(\psi_{p0} - \psi_0\right)}{\partial z^2} + D\left(\psi_{p2} - \psi_2\right)\right]$$
$$= \left(1-q\right)\left[D\frac{\partial\psi_0}{\partial t} + \frac{1}{r}\frac{\partial\psi_1}{\partial z}\left(\frac{\partial^3\psi_0}{\partial r^3} - \frac{3}{r}\frac{\partial^2\psi_0}{\partial r^2} + \frac{3}{r^2}\frac{\partial\psi_0}{\partial r}\right)\right.$$
$$\left. + \frac{1}{r}\frac{\partial\psi_0}{\partial z}\left(\frac{\partial^3\psi_1}{\partial r^3} - \frac{3}{r}\frac{\partial^2\psi_1}{\partial r^2} + \frac{3}{r^2}\frac{\partial\psi_1}{\partial r}\right) - \frac{1}{r}\frac{\partial\psi_1}{\partial r}D\frac{\partial\psi_0}{\partial z} - \frac{1}{r}\frac{\partial\psi_0}{\partial r}D\frac{\partial\psi_1}{\partial z}\right]. \qquad ...(25)$$

(b) *For the particle phase*:
Substituting equation (22) into equation (17) gives:

$$q\chi_p D\left(\psi_0 - \psi_{p0}\right) = 0. \qquad ...(26)$$

$$q\chi_p D\left(\psi_1 - \psi_{p1}\right) = q\left[D\frac{\partial\psi_{p0}}{\partial t} + \frac{1}{r}\frac{\partial\psi_{p0}}{\partial z}\left(\frac{\partial^3\psi_{p0}}{\partial r^3} - \frac{3}{r}\frac{\partial^2\psi_{p0}}{\partial r^2} + \frac{3}{r^2}\frac{\partial\psi_{p0}}{\partial r}\right)\right.$$
$$\left. - \frac{1}{r}\frac{\partial\psi_{p0}}{\partial r}D\frac{\partial\psi_{p0}}{\partial z}\right]. \qquad ...(27)$$

$$q\chi_p\left[D\left(\psi_2 - \psi_{p2}\right) + \frac{\partial^2\left(\psi_0 - \psi_{p0}\right)}{\partial z^2}\right]$$
$$= q\left[D\frac{\partial\psi_{p0}}{\partial t} + \frac{1}{r}\frac{\partial\psi_{p1}}{\partial z}\left(\frac{\partial^3\psi_{p0}}{\partial r^3} - \frac{3}{r}\frac{\partial^2\psi_{p0}}{\partial r^2} + \frac{3}{r^2}\frac{\partial\psi_{p0}}{\partial r}\right)\right.$$
$$+ \frac{1}{r}\frac{\partial\psi_{p0}}{\partial z}\left(\frac{\partial^3\psi_{p1}}{\partial r^3} - \frac{3}{r}\frac{\partial^2\psi_{p1}}{\partial r^2} + \frac{3}{r^2}\frac{\partial\psi_{p1}}{\partial r}\right)$$
$$\left. - \frac{1}{r}\frac{\partial\psi_{p1}}{\partial r}D\frac{\partial\psi_{p0}}{\partial z} - \frac{1}{r}\frac{\partial\psi_{p0}}{\partial r}D\frac{\partial\psi_{p1}}{\partial z}\right]. \qquad ...(28)$$

Where χ_f and χ_p are as defined above;

$$D \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} \equiv r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right); \text{ and } L \equiv \frac{\partial^4}{\partial r^4} - \frac{2}{r}\frac{\partial^3}{\partial r^3} + \frac{3}{r^2}\frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r}$$

Under this expansion therefore, the corresponding boundary conditions become:

At $r = \eta = 1 + \in \cos\beta$; such that $\beta = 2\pi(z-t)$;

$$\left.\begin{aligned}
&\frac{\partial\psi_0}{\partial r} = \frac{\partial\psi_1}{\partial r} = \frac{\partial\psi_2}{\partial r} = 0; && \text{(i)}\\
&\frac{\partial\psi_0}{\partial z} = \frac{\partial\psi_{p0}}{\partial z} = 2\pi \in r\sin\beta; && \text{(ii)}\\
&\frac{\partial\psi_1}{\partial z} = \frac{\partial\psi_{p1}}{\partial z} = 0; && \text{(iii)}\\
&\frac{\partial\psi_2}{\partial z} = \frac{\partial\psi_{p0}}{\partial z} = 0; && \text{(iv)}
\end{aligned}\right\} \qquad ...(29)$$

For the Axial Pressure Gradient:

Substituting equations (21) and (22) into the dimensionless equations (18) and (19), respectively; collecting and equating coefficients of equal powers of δ on both sides of the respective equations a long wave approximation for the axial pressure gradient can be obtained in the form:

$$\frac{\partial p}{\partial z} = \frac{\partial p_0}{\partial z} + \delta\frac{\partial p_1}{\partial z} + \delta^2\frac{\partial p_2}{\partial z} + \cdot\ \cdot\ \cdot\ . \qquad ...(30)$$

(a) *For the fluid phase*:

From equation (18), substituting equation (21) and comparing with equation (30) gives the following sequences of equations:

$$(1-q)\frac{\partial p_0}{\partial z} = \frac{1}{Re}\left(-\frac{1}{r^3}\frac{\partial\psi_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi_0}{\partial r^2} - \frac{1}{r}\frac{\partial^3\psi}{\partial r^3}\right) - q\chi_f\frac{1}{r}\frac{\partial(\psi_{p0}-\psi_0)}{\partial r}. \qquad ...(31)$$

$$(1-q)\frac{\partial p_1}{\partial z} = \frac{1}{r}\frac{\partial^2\psi_0}{\partial r\partial t} - \frac{1}{r^3}\frac{\partial\psi_0}{\partial z}\frac{\partial\psi_0}{\partial r} + \frac{1}{r^2}\frac{\partial\psi_0}{\partial z}\frac{\partial^2\psi_0}{\partial r^2} - \frac{1}{r^2}\frac{\partial\psi_0}{\partial r}\frac{\partial^2\psi_0}{\partial z\partial r}$$
$$+\frac{1}{Re}\left(-\frac{1}{r^3}\frac{\partial\psi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi_1}{\partial r^2} - \frac{1}{r}\frac{\partial^3\psi}{\partial r^3}\right) - q\chi_f\frac{1}{r}\frac{\partial(\psi_{p1}-\psi_1)}{\partial r}. \qquad ...(32)$$

$$(1-q)\frac{\partial p_2}{\partial z} = \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial t} - \frac{1}{r^3}\left(\frac{\partial\psi_0}{\partial z}\frac{\partial\psi_1}{\partial r} + \frac{\partial\psi_1}{\partial z}\frac{\partial\psi_0}{\partial r}\right)$$

$$+\frac{1}{r^2}\left(\frac{\partial\psi_1}{\partial z}\frac{\partial^2\psi_0}{\partial r^2}+\frac{\partial\psi_0}{\partial z}\frac{\partial^2\psi_1}{\partial r^2}\right)-\frac{1}{r^2}\left(\frac{\partial\psi_0}{\partial r}\frac{\partial^2\psi_1}{\partial z\partial r}+\frac{\partial\psi_1}{\partial r}\frac{\partial^2\psi_0}{\partial z\partial r}\right)$$

$$+\frac{1}{Re}\left(-\frac{1}{r^3}\frac{\partial\psi_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi_2}{\partial r^2}-\frac{1}{r}\frac{\partial^3\psi_2}{\partial r^3}-\frac{1}{r}\frac{\partial^3\psi_2}{\partial^2 z\partial r}\right)-q\chi_f\frac{1}{r}\frac{\partial\left(\psi_{p2}-\psi_2\right)}{\partial r}. \quad ...(33)$$

(b) *For the particle phase*:

From equation (19), substituting equation (22) and comparing with equation (30) gives the following sequences of equations:

$$q\frac{\partial p_0}{\partial z}=-q\chi_f\frac{1}{r}\frac{\partial\left(\psi_0-\psi_{p0}\right)}{\partial r} \quad ...(34)$$

$$q\frac{\partial\psi_1}{\partial z}=q\frac{\rho_p}{\rho_f}\left(\frac{1}{r}\frac{\partial^2\psi_{p0}}{\partial r\partial t}-\frac{1}{r^3}\frac{\partial\psi_{p0}}{\partial z}\frac{\partial\psi_{p0}}{\partial r}+\frac{1}{r^2}\frac{\partial\psi_{p0}}{\partial z}\frac{\partial^2\psi_{p0}}{\partial r^2}-\frac{1}{r^2}\frac{\partial\psi_{p0}}{\partial r}\frac{\partial^2\psi_{p0}}{\partial z\partial r}\right)$$

$$-q\chi_f\frac{1}{r}\frac{\partial\left(\psi_1-\psi_{p1}\right)}{\partial r}. \quad ...(35)$$

$$q\frac{\partial p_2}{\partial z}=q\frac{\rho_p}{\rho_f}\left[\frac{1}{r}\frac{\partial^2\psi_{p1}}{\partial r\partial t}-\frac{1}{r^3}\left(\frac{\partial\psi_{p0}}{\partial z}\frac{\partial\psi_{p1}}{\partial r}+\frac{\partial\psi_{p1}}{\partial z}\frac{\partial\psi_{p0}}{\partial r}\right)\right.$$

$$+\frac{1}{r^2}\left(\frac{\partial\psi_{p1}}{\partial z}\frac{\partial^2\psi_{p0}}{\partial r^2}+\frac{\partial\psi_{p0}}{\partial z}\frac{\partial^2\psi_{p1}}{\partial r^2}\right)$$

$$\left.-\frac{1}{r^2}\left(\frac{\partial\psi_{p0}}{\partial r}\frac{\partial^2\psi_{p1}}{\partial z\partial r}+\frac{\partial\psi_{p1}}{\partial r}\frac{\partial^2\psi_{p0}}{\partial z\partial r}\right)\right]$$

$$-q\chi_f\frac{1}{r}\frac{\partial\left(\psi_2-\psi_{p2}\right)}{\partial r}. \quad ...(36)$$

RESULTS

The sequences of the differential equations are solved in stages to obtain values of ψ and ψ_p, and consequently the other flow parameters; the axial flow velocities, volume flux and axial pressure gradient. The various stages are termed zeroth, first and second order approximations, with suffixes 0, 1 and 2. The results so obtained are further investigated for time-averaged flow quantities.

In this study however, we restrict to the zeroth order solutions only. These are the solutions that represent the limiting case of $\delta\to 0$. Since the distance z is measured by a scale of wavelength λ, the complete wave structure is retained in the boundary conditions when the limit process $\delta \to 0$, with $(r, z, t;\ \varepsilon, Re)$ fixed, is taken. Hence "the zeroth order solutions represent meaningful limiting solutions for very long waves". Li, (1970). "The zeroth order solutions are more applicable to physical problems". Muhammad and Sesay, (2010).

(a) *For the fluid phase:*

From equation (26), we have $q\chi_p D(\psi_0 - \psi_{p0}) = 0 \Rightarrow D(\psi_0 - \psi_{p0}) = 0$. Substituting into equation (23) gives: $\frac{1}{Re} L\psi_0 = 0$; or $L\psi_0 = 0$,

$$\Rightarrow \left(\frac{\partial^4}{\partial r^4} - \frac{2}{r}\frac{\partial^3}{\partial r^3} + \frac{3}{r^2}\frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} \right)\psi_0 = 0$$

The general solution will thus be: $\psi_0 = A(z,t)r^4 + B(z,t)r^3 + C(z,t)r^2 + E(z,t)r$,

Where A, B, C, E are constants of integration.

The boundary conditions and the symmetry conditions would demand that ψ_0 be an even function in r. Hence $B(z,t) \equiv E(z,t) = 0$. This gives a general solution of the form:

$$\psi_0 = A_1(z,t)r^4 + A_2(z,t)r^2. \qquad ...(37)$$

Where A_1 and A_2 which are constants of integration which, as presented in Li, (1970), on applying the boundary conditions, are found to satisfy the following simultaneous equations:

$$\left.\begin{aligned} &\eta^3 \frac{\partial A_1}{\partial z} + \eta \frac{\partial A_2}{\partial z} = 2\pi\varepsilon \sin\beta && (i) \\ &2\eta^2 A_1 + A_2 = 0 && (ii) \end{aligned}\right\} \qquad ...(38)$$

Solving the simultaneous equations (38) gives:

$$\left.\begin{aligned} &A_1 = \frac{1}{\eta^4}\left(C(t) + \frac{1}{4}\varepsilon^2 + \varepsilon\cos\beta + \frac{1}{4}\varepsilon^2\cos 2\beta \right) && (i) \\ &A_2 = -\frac{2}{\eta^2}\left(C(t) + \frac{1}{4}\varepsilon^2 + \varepsilon\cos\beta + \frac{1}{4}\varepsilon^2\cos 2\beta \right) && (ii) \end{aligned}\right\} \qquad ...(39)$$

The arbitrary function of time, C(t), arises from integrating with respect to z. It is related to the volumetric flow, hence will be determined later.

The **instantaneous axial velocity**, v_0, can be obtained from equation (37) $\left(recall:\ v = -\frac{1}{r}\frac{\partial \psi}{\partial r} \right)$. $\therefore \quad v_0 = -4A_1 r^2 - 2A_2$ ...(40)

Substituting for A_1 and A_2 gives:

$$v_0 = \frac{4}{\eta^2}\left(C(t) + \frac{1}{4}\varepsilon^2 + \varepsilon\cos\beta + \frac{1}{4}\varepsilon^2\cos 2\beta \right)\left(1 - \frac{r^2}{\eta^2} \right). \quad ...(41)$$

To find expression for **the axial pressure gradient**, we add equations (31) and (34) to obtain:

$$\frac{\partial p_0}{\partial z} = \frac{1}{Re}\left(-\frac{1}{r^3}\frac{\partial \psi_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi_0}{\partial r^2} - \frac{1}{r}\frac{\partial^3 \psi}{\partial r^3} \right). \quad ...(42)$$

Substituting for ψ_0 as obtained in equation (37) gives: $\frac{\partial p_0}{\partial z} = -\frac{1}{Re}16A_1$. ...(43)

$$\Rightarrow \quad \frac{\partial p_0}{\partial z} = \frac{1}{Re}\frac{16}{\eta^4}\left(C(t) + \frac{1}{4}\varepsilon^2 + \varepsilon\cos\beta + \frac{1}{4}\varepsilon^2\cos 2\beta \right) \quad ...(44)$$

(b) *For the particle phase*:

Starting from equation (26) gives : $D\left(\psi_0 - \psi_{p0}\right) = 0$.

$$\Rightarrow \quad r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r} \right)\left(\psi_0 - \psi_{p0}\right) = 0 \text{; or } -\frac{1}{r}\frac{\partial}{\partial r}\left(\psi_0 - \psi_{p0}\right) = k_p. \quad ...(45)$$

Following the symmetry condition as mentioned earlier, we have:

$$\psi_{p0} = \psi_0 - \frac{1}{2}k_p r^2. \quad ...(46)$$

Also, from equation (45);

$$-\frac{1}{r}\frac{\partial \psi_0}{\partial r} + \frac{1}{r}\frac{\partial \psi_{p0}}{\partial r} = k_p \text{; then } v_0 - v_{p0} = k_p \text{. Thus: } v_{p0} = v_0 - k_p. \quad ...(47)$$

To obtain k_p, equation (45) is substituted into equation (34), which gives:

$$k_p = -\frac{16}{Re\chi_f}A_1; \quad ...(48)$$

$$\text{Or } k_p = -\frac{16}{Re\chi_f}\frac{1}{\eta^4}\left[C(t) + \frac{1}{4}\varepsilon^2 + \varepsilon\cos\beta + \frac{1}{4}\varepsilon^2\cos 2\beta \right]. \quad ...(49)$$

However, it should be noted that k_p is purely a function of t only. This can be shown below:

$$\frac{\partial \psi_{p0}}{\partial z} = \frac{\partial \psi_0}{\partial z} - \frac{\partial k_p}{\partial z} r \,;$$

From boundary conditions (29) (ii); $\frac{\partial k_p}{\partial z}\eta = 0$, thus $\frac{\partial k_p}{\partial z} = 0$.

Integrating with respect to z therefore, gives: $k_p = k_p(t)$. Henceforth it will be represented as such.

Differentiating equation (46) with respect to z gives:

To find expression for the instantaneous axial velocity for the particle phase, we consider equations (40) and (47), from which we have:

$$v_{p0} = v_0 - k_p; \quad \text{hence } v_{p0} = -4A_1 r^2 - 2A_2 - k_p \qquad ...(50)$$

The instantaneous volume flow rate, which is usually referred to as volume flux, denoted by Q_0 is defined as:

Substituting for ψ_{p0} as obtained in equation (46) gives:

$$Q_0 = -2\pi\left[(1-q)\psi_0 + q\left(\psi_0 - \frac{1}{2}k_p(t)r^2\right)\right]\Bigg|_{\mathbf{r}=\boldsymbol{\eta}} .$$

$$= -2\pi\left[\psi_0 - \frac{1}{2}qk_p(t)r^2\right]\Bigg|_{\mathbf{r}=\boldsymbol{\eta}} .$$

Substituting for ψ_0 as obtained in equation (37) gives:

$$Q_0 = -2\pi\left[A_1 r^4 + A_2 r^2 - \frac{1}{2}qk_p(t)r^2\right]\Bigg|_{\mathbf{r}=\boldsymbol{\eta}} .$$

Thus: $$Q_0 = -2\pi\eta^2\left[A_1\eta^2 + A_2 - \frac{1}{2}qk_p(t)\right]. \qquad ...(51)$$

Substituting for the values of A_1 and A_2 as obtained in equation (39) gives:

$$Q_0 = 2\pi\left(C(t) + \frac{1}{4}\varepsilon^2 + \varepsilon\cos\beta + \frac{1}{4}\varepsilon^2\cos 2\beta\right) + q\pi k_p(t)\eta^2 . \qquad ...(52).$$

Before proceeding to the next stage, let us summarize the zeroth-order flow quantities so far obtained;

(i) The instantaneous axial velocity for the fluid phase (equation (41)):

$$\mathbf{v_0 = \frac{4}{\eta^2}\left(C(t) + \frac{1}{4}\varepsilon^2 + \varepsilon\cos\beta + \frac{1}{4}\varepsilon^2\cos2\beta\right)\left(1 - \frac{r^2}{\eta^2}\right)}.$$

(ii) The instantaneous axial velocity for the particle phase (from equation (50)):

$$\mathbf{v_{p0} = v_0 - k_p(t)};\ \text{where}\ \mathbf{k_p}(t) = -\frac{\mathbf{16}}{\mathbf{Re\chi_f}}\frac{\mathbf{1}}{\mathbf{\eta^4}}\left[\mathbf{C(t) + \frac{1}{4}\varepsilon^2 + \varepsilon\cos\beta + \frac{1}{4}\varepsilon^2\cos2\beta}\right].$$

(iii) The axial pressure gradient, which is common for the whole suspension:

$$\mathbf{\frac{\partial p_0}{\partial z} = \frac{1}{Re}\frac{16}{\eta^4}\left(C(t) + \frac{1}{4}\varepsilon^2 + \varepsilon\cos\beta + \frac{1}{4}\varepsilon^2\cos2\beta\right)}$$

(iv) The instantaneous volume flow rate or volume flux, which is common for the whole suspension as given in equation (52):

$$\mathbf{Q_0 = 2\pi\left(C(t) + \frac{1}{4}\varepsilon^2 + \varepsilon\cos\beta + \frac{1}{4}\varepsilon^2\cos2\beta\right) + q\pi k_p(t)\eta^2}.$$

It can be noticed that the arbitrary constant of integration $C(t)$ is appearing in all the expressions. Its value will obviously depend on the type of study model. From Li, (1970), "it relates the axial pressure gradient to the instantaneous volume flow rate. For the special case, $C(t) \equiv 0$, the flow is due solely to peristalsis, because $v_0 = 0$ if $\varepsilon = 0$." Also, from Muhammad and Sesay, (2010), where the case of pure (natural) peristaltic flow, with no initially applied external pressure gradient, is strictly being considered, it is obvious that $C(t) = 0$. This is because, if $\varepsilon = 0$, a case of no amplitude, then $v_0 = 0$. On the other hand, if $C(t) \neq 0$, it implies that there is an applied external pressure gradient, which contradicts the earlier assumption of a pure peristaltic flow from an initially stagnant situation, because it will imply that even if $\varepsilon = 0$, there will still be fluid velocity, such that $v_0 \neq 0$, which is a contrary case."

However, since this study is strictly based on Li's (1970) model, "the case of $C(t) \equiv C_0$, a pure constant, is considered. This corresponds to the case of imposing a constant volume flux through the tube in addition to that due to peristalsis."

Time-Averaged Flow Quantities

We will need to investigate the behavior of the results of the respective flow quantities averaged over one period of the wave motion. By this, we are kind of considering some particular constant volume flux of the flow. The mean flow quantities averaged over one period are obtained by integrating with respect to time over one period of wave motion.

Recall that: $\eta = 1 + \varepsilon \cos 2\pi (z - t)$.

Hence, by Spiegel, (1968), Zien and Ostrach, (1969), and Li, (1970) methods we have:

$$\int_0^1 \frac{dt}{\eta} = \left(1 - \varepsilon^2\right)^{-\frac{1}{2}} .$$

$$\int_0^1 \frac{dt}{\eta^2} = \frac{1}{2}\left(2 + \varepsilon^2\right)\left(1 - \varepsilon^2\right)^{-\frac{3}{2}} .$$

$$\int_0^1 \frac{dt}{\eta^3} = \frac{1}{2}\left(2 + \varepsilon^2\right)\left(1 - \varepsilon^2\right)^{-\frac{5}{2}}$$

$$\int_0^1 \frac{dt}{\eta^4} = \frac{1}{2}\left(2 + \varepsilon^2\right)\left(1 - \varepsilon^2\right)^{-7/2}$$

$$\int_0^1 \int_0^1 \frac{\cos 2\pi (z-t)}{\eta} dt = \frac{1}{\varepsilon}\left[1 - \left(1 - \varepsilon^2\right)^{-\frac{1}{2}}\right]$$

$$\int_0^1 \frac{\cos 2\pi (z-t)}{\eta^2} dt = \frac{1}{2}\varepsilon\left(1 - \varepsilon^2\right)^{-\frac{3}{2}} .$$

$$\int_0^1 \frac{\cos 2\pi (z-t)}{\eta^3} dt = \frac{3}{2}\varepsilon(1 - \varepsilon^2)^{-\frac{5}{2}}$$

$$\int_0^1 \frac{\cos 2\pi (z-t)}{\eta^4} dt = \frac{5}{2}\varepsilon\left(1 - \varepsilon^2\right)^{-7/2} .$$

Thus, $\overline{v_0}$, the mean axial velocity for the fluid phase is defined as: $\overline{v_0} = \int_0^1 v_0 dt$.

$$\Rightarrow \overline{v_0} = 2\left\{1 - \left(1 - \varepsilon^2\right)^{-\frac{3}{2}} + \left[\left(1 + \frac{3}{2}\varepsilon^2\right)\left(1 - \varepsilon^2\right)^{-\frac{7}{2}} - \left(1 - \varepsilon^2\right)^{-\frac{3}{2}}\right] r^2 + C_0\left(1 - \varepsilon^2\right)^{-\frac{3}{2}}\left[2 - \left(2 + 3\varepsilon^2\right)\left(1 - \varepsilon^2\right)^{-2} r^2\right]\right\} . \quad ...(53)$$

The mean axial velocity for the particle phase is defined as: $\bar{v}_{p0} = \int_0^1 v_{p0} dt$.

$$\Rightarrow \bar{v}_{p0} = \frac{1}{16}\left(\frac{16}{\chi_f Re} - 4r^2\right)\left(1 - \varepsilon^2\right)^{-\frac{7}{2}}\left\{\left(8C_0 + 2\varepsilon^2\right)\left(2 + 3\varepsilon^2\right) + \left(8 + \varepsilon\right)\left[\left(2 + \frac{\varepsilon^2}{2}\right)\left(1 - \varepsilon^2\right)^2 - \left(2 + 3\varepsilon^2\right)\right]\right\} + \frac{1}{2}\left(1 - \varepsilon^2\right)^{-\frac{3}{2}}\left[8C_0 - 6\varepsilon^2\left(1 + \varepsilon\right)\right] . \quad ...(54)$$

The mean axial pressure gradient for the whole suspension (which is common for both phases) is defined as: $\frac{\overline{\partial p_0}}{\partial z} = \int_0^1 \frac{\partial p_0}{\partial z} dt$.

$$\Rightarrow \frac{\overline{\partial p_0}}{\partial z} = \frac{4}{Re}\left[\varepsilon^2\left(7-2\varepsilon^2\right)-2C_0\left(2+3\varepsilon^2\right)\right]\left(1-\varepsilon^2\right)^{-\frac{7}{2}}\Bigg]. \quad ...(55)$$

The mean volume flux for the whole suspension (which is also common for both phases) is defined as: $\overline{Q}_0 = \int_0^1 Q_0 dt$.

$$\Rightarrow \overline{Q}_0 = -2\pi\left(C_0+\frac{\varepsilon^2}{4}\right)-q\frac{16\pi}{\chi_f Re}\left[C_0+\frac{\varepsilon^2}{8}\left(1+\varepsilon\right)\left(1-\varepsilon^2\right)^{-\frac{1}{2}}\right]. \quad ...(56)$$

The above equation can be re-expressed as:

$$\overline{Q}_0 = -2\left\{\pi\left[1+q\frac{8}{\chi_f Re}\left(1-\varepsilon^2\right)^{-\frac{1}{2}}\right]C_0+\frac{\varepsilon^2}{4}+q\frac{1}{\chi_f Re}\varepsilon^2\left(1+\varepsilon\right)\left(1-\varepsilon^2\right)^{-\frac{1}{2}}\right\} \quad ....(57)$$

From which we have:

$$C_0 = -\left[1+8qS_0\left(1-\varepsilon^2\right)^{-\frac{1}{2}}\right]^{-1}\left[\frac{Q_0}{2\pi}+\frac{\varepsilon^2}{4}+qS_0\varepsilon^2\left(1+\varepsilon\right)\left(1-\varepsilon^2\right)^{-\frac{1}{2}}\right]. \quad ...(58)$$

Where $S_0 = \frac{1}{\chi_f Re}$.

Substituting the value of C_0, the mean axial velocities and pressure gradients can be expressed in terms of the mean flux Q_0, as follows

:

$$\overline{u}_0 = 2\left\{1-\left(1-\varepsilon^2\right)^{-\frac{3}{2}}+\left[\left(1+\frac{3}{2}\varepsilon^2\right)\left(1-\varepsilon^2\right)^{-\frac{7}{2}}-\left(1-\varepsilon^2\right)^{-\frac{3}{2}}\right]r^2\right.$$

$$-\frac{\left[\frac{Q_0}{2\pi}+\frac{\varepsilon^2}{4}+qS_0\varepsilon^2(1+\varepsilon)(1-\varepsilon^2)^{-\frac{1}{2}}\right]}{\left[1+8qS_0(1-\varepsilon^2)^{-\frac{1}{2}}\right]}(1-\varepsilon^2)^{-\frac{3}{2}}\left[2-(2+3\varepsilon^2)(1-\varepsilon^2)^{-2}\right]r^2\Bigg\} \ldots(59)$$

$$\bar{u}_{p0}=\left(\frac{1}{\chi_f Re}-\frac{1}{4}r^2\right)(1-\varepsilon^2)^{-\frac{7}{2}}\Bigg\{(8+\varepsilon)\left[\left(2+\frac{\varepsilon^2}{2}\right)(1-\varepsilon^2)^2-(2+3\varepsilon^2)\right]$$

$$+(2+3\varepsilon^2)\left[2\varepsilon^2-\frac{8\left[\frac{Q_0}{2\pi}+\frac{\varepsilon^2}{4}+qS_0\varepsilon^2(1+\varepsilon)(1-\varepsilon^2)^{-\frac{1}{2}}\right]}{\left[1+8qS_0(1-\varepsilon^2)^{-\frac{1}{2}}\right]}\right]\Bigg\}$$

$$-\frac{1}{2}(1-\varepsilon^2)^{-\frac{3}{2}}\left[6\varepsilon^2(1+\varepsilon)+\frac{8\left[\frac{Q_0}{2\pi}+\frac{\varepsilon^2}{4}+qS_0\varepsilon^2(1+\varepsilon)(1-\varepsilon^2)^{-\frac{1}{2}}\right]}{\left[1+8qS_0(1-\varepsilon^2)^{-\frac{1}{2}}\right]}\right] \ldots(60)$$

$$\frac{\partial\bar{p}_0}{\partial z}=\frac{4}{Re}\left\{\varepsilon^2(7-2\varepsilon^2)+\frac{2(2+3\varepsilon^2)\left[\frac{Q_0}{2\pi}+\frac{\varepsilon^2}{4}+qS_0\varepsilon^2(1+\varepsilon)(1-\varepsilon^2)^{-\frac{1}{2}}\right]}{\left[1+8qS_0(1-\varepsilon^2)^{-\frac{1}{2}}\right]}\right\}(1-\varepsilon^2)^{-\frac{7}{2}}$$

...(61)

DISCUSSION

Observing the various results obtained for the fluid phase of the suspension, the results agree with those of Li, (1970), with slight modification, as would be expected. It is obvious that the modification arises due to the interacting term between the particle and fluid phases. Equations (41) and (50), confirmed the assertion that the axial flow velocity is directly proportional to the square of the radius of the flow tract (tube/pipe). This is so, even for the particle phase (equation (50)).

It is interesting to note that, in all the results obtained, (equations (41) – (50)); even for the case of time-averaged flow quantities (equations (53) – (57)); except for the volume flux, equations (52) and (57); are independent of the particle volume fraction, q. It is clear that if the volume fraction, q, if made zero, it becomes a clear case of single-phase fluid. Hence one can conclude from this study that, the fluid type, as long as it is regarded as a Newtonian fluid, does not adversely affect the outcome of results. That is to say that any suspension, as described in this study, may conveniently be considered as a uniform single-phase fluid with little effect on the flow quantities, the drag effect of the suspended particles notwithstanding.

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