SPLINE INTERPOLATION METHOD OF SOLVING SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM USING POLYNOMIAL AND NON-POLYNOMIAL SPLINES WITH ERROR COMPARISON

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ABSTRACT

We consider two polynomial Spline methods to calculate the numerical solution as well as the maximum absolute errors of a singularly perturbed two point boundary value problems. The two methods are linear and nonlinear polynomial spline and nonlinear polynomial Spline tends to give accurate, stable and consistent results compared with the linear polynomial spline. Also interms of the formulation of the two methods, the nonlinear polynomial spline is in terms of differential equation of order two. The goal is to find the maximum absolute errors between the liner polynomial and nonlinear polynomial spline methods in relation to the exact solution. The applications of these splines to singularly perturbed two point boundary value problem resulted to linear algebraic system of equations which are then solved by Gaussian elimination method to obtain the unknown constants arising from the spline used.

Keywords: Singularly Perturbation, Polynomial and Non-polynomial Splines and Maximum Absolute Error.

INTRODUCTION

Differential equations are useful in describing mathematical models for various physical processes while there are many theoretical results on existence, uniqueness and properties of solutions of such equations, usually only the simplest specific problems can be solve explicitly when the nonlinear terms are involved and we usually construct approximate solutions. Since only limited classes of the equations are solved by analytical means, numerical solution of these differential equations is of practical importance. Polynomials have long been the functions most widely used to approximate other functions mainly because of their simple mathematical properties. However, it is well-known that polynomials of high degree tend to oscillate strongly and in many cases they liable to produce very poor approximations. The spline function can integrate and differentiated due to being piecewise polynomial and can easily stored and implemented on digital computers. Thus, spline functions are adopted to numerical methods to get the solution of the differential equations. Numerical methods with spline functions in getting the approximate solution of the differential equations lead to band matrices which are solvable easily with algorithms having low cost of computation.

According to Schumaker (2007), the arpid development of spline function is primarily due to their great usefulness in applications. Classes of spline function possess may nice structural properties. This makes splines desirable over polynomial, piecewise polynomial and some other approximating functions. The theory of spline functions had a rather modest development until 1960. Early contributors to this development include deBoor (1962, 1963) and Schoenberg (1964). Some of the earliest papers using spline functions for smooth approximate solution of ordinary and partial differential equations include Albsing and Hoskin's (1996), Feyfe (1977) and Sastry (1976). Recently, non-polynomial splines were used for numerical solution of obstacle problems in Islam et al. (2006). Third order BVPs have been treated in Islam et al (2007) and fourth order equations are discussed in Taiwo and Ogunlaran (2011). We are interested in finding the numerical solution as well as maximum absolute error between linear polynomial spline and nonlinear polynomial spline.

Spline methods

This is the formulation of a Spline function approximation and the development of some methods for numerical solution of second-order singularly perturbed boundary value problems;

We derived the methods to solve a boundary value problem in the finite interval$[a,b]$ and partition the interval using equally spaced knots$x_i = a + ih, i = 0,1,2,\dots,N,\ x_0 = a,\ x_N = b,\ h = \frac{(b-a)}{N}$, where N is an arbitrary positive integer. For each sub-intervalx_i, x_{i+1}, $i = 0,1,2,\dots,N-1$.

Linear Polynomial Spline Function

The polynomial Spline function has the form:

$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$ (1)

Where$i = 0,1,2,\dots,N-1$

and$a_i b_i c_i$ and d_i are constants.

A polynomial function $S(x)$ of class $C^2[a,b]$ interpolates $y(x)$ at the grid point x_i for $i = 0,1,2,\dots,N$. Let y_i be an approximation to $y(x_i)$, obtained by the polynomial spline S passing through the points x_i, y_i and $(x_{i+1},\ y_{i+1})$. The spline functionEq.(1) is not only required to satisfy the given differential equation and the associated boundary conditions at x_i and x_{i+1}, but also satisfy the continuity of first derivatives at the common nodes (x_i, y_i). We derived an expression for the coefficients of Eq.(1) in-terms of y_i, y_{i+1}, M_i and M_{i+1} thus,

$S_{(x_i)} = y_i, S_{(x_{i+1})} = y_{i+1}, S''(x_i) = M_i, S''(x_{i+1}) = M_{i+1}$ (2)

From the algebraic simplification, we obtain as from Eq.(2)

$a_i = y_i, b_i = \frac{(y_{i+1}-y_i)}{h} - \frac{h(M_{i+1}+2M_i)}{6}, c_i = \frac{M_i}{2}, d_i = \frac{M_{i+1}-M_i}{6h}$ (3)

Where $i = 0,1,2,\dots,N-1$.

One sided limit of the derivative of $S(x)$ are obtain as

$S'(x_i^-) = \frac{1}{h}(y_i - y_{i-1}) + \frac{h}{6}(M_i - 2M_{i-1}),$ (4)

Where $i = 0,1,2,\dots,N$.

and

$S'(x_i^+) = \frac{1}{h}(y_{i+1} - y_i) - \frac{h}{6}(M_{i+1} + 2M_i),$ (5)

Where $i = 0,1,2,\dots,N-1$.

Using the continuity condition of the first derivatives at x_i, y_i, that is

$S'_{i-1}(x_i) = S'_i(x_i)$, we obtain the following consistency relation:

$M_{i+1} + 4M_i + M_{i-1} = \frac{6}{h^2}(y_{i+1} - 2y_i + y_{i-1}),$ (6)

Where $i = 1, 2, \ldots, N-1$.

Singularly Perturbed Boundary Value Problem

We consider second order singularly perturbed boundary value problem of the form:

$-\epsilon y'' + q(x)y' + p(x)y = f(x)$ (7)

$y(0) = A, y(1) = B$ (8)

Where A, B are constants and ϵ is a small positive parameter such that $0 < \epsilon \leq 1$ and $q(x), p(x), f(x)$ are small bounded real functions.

Linear Polynomial

The methods developed for the solution of the boundary value problem Eq.(7) are based on the spline approximation function discussed (1) . The interval [0,1] is partitioned into a set of N equal sub-interval with length $h = \frac{1}{N}$, such that the nodal point $x_0 = 0, x_N = 1$ and $x_i = ih,\ i = 1(1)N-1$. Let $y_i$been approximation to (x_i) , obtained by the segment $S(x)$ of the spline function passing through the points (x_i, y_i) and $(x_{i+1}y_{i+1})$, where $S(x)$ is as defined by Eq.(1).

We considered two cases of Eq. (7)

Case (a): when $q(x) = 0$

Now, setting $x = x_i$, in Eq. (7) and making use of Eq. (2), we have

$M_i = \frac{p(x_i)}{\epsilon} y_i - \frac{1}{\epsilon} f_i$ (9)

It follows that

$M_{i-1} = \frac{p(x_{i-1})}{\epsilon} y_{i-1} - \frac{1}{\epsilon} f_{i-1}$ (10)

and

$M_{i+1} = \frac{p(x_{i+1})}{\epsilon} y_{i+1} - \frac{1}{\epsilon} f_{i+1}$ (11)

Using Eq.(9), Eq.(10) and Eq.(11) in the consistency relation Eq.(6), we obtain

$(h^2 p_{i-1} - 6\epsilon)y_{i-1} + 4(h^2 p_i + 3\epsilon)y_i + (h^2 p_{i+1} - 6\epsilon)y_{i+1} = h^2(f_{i-1} + 4f_i + f_{i+1})$ (12)

Where $i = 1, 2, \ldots, N-1$.

Thus, Eq.(12) together with the two boundary conditions Eq.(8) gives rise to a tridiagonal set of $(n+1)$ algebraic equations which are solved by Gaussian elimination method for the $(n+1)$ unknown $y_i, i = 0, 1, 2, \ldots, N$.

Case (b): when $q(x) \neq 0$

Now, setting $x = x_i$ in Eq. (7) and making use of Eq. (2)

$M_i = \frac{q(x_i)}{\epsilon} y'_i + \frac{p(x_i)}{\epsilon} y_i - \frac{1}{\epsilon} f_i$ (13)

It follows that

$M_{i-1} = \frac{q(x_{i-1})}{\epsilon} y'_{i-1} + \frac{p(x_{i-1})}{\epsilon} y_{i-1} - \frac{1}{\epsilon} f_{i-1}$ (14)

$M_{i+1} = \frac{q(x_{i+1})}{\epsilon} y'_{i+1} + \frac{p(x_{i+1})}{\epsilon} y_{i+1} - \frac{1}{\epsilon} f_{i+1}$ (15)

Substituting Eq. (13), Eq.(14) and Eq.(15) in the consistency relation Eq.(6) and using the following approximations for finite difference of two order derivatives of y yields;

$y'_i = \frac{y_{i+1} - y_{i-1}}{2h} + o(h^2)$

$y'_{i+1} = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} + o(h^2)$

$$y'_{i-1} = \frac{-y_{i+1}+4y_i-3y_{i-1}}{2h} + o(h^2) \qquad (16)$$

We obtain.

$$\left(6\epsilon + \frac{p_{i+1}}{2h} - \frac{2p_i}{h} + q_{i-1} - \frac{3p_{i-1}}{2h}\right) y_{i-1} + \left(-12\epsilon - \frac{2p_{i-1}}{h} + 4q_{i-1} + \frac{2p_{i-1}}{h}\right) y_i + \left(6\epsilon + \frac{3p_{i+1}}{2h} + \frac{2p_i}{h} + q_{i+1} - \frac{p_{i-1}}{2h}\right) y_{i+1} = h^2(f_{i-1} + 4f_i + f_{i+1})$$

Where $i = 1, 2, \dots, N-1$ (17)

Thus, Eq.(17) together with the two boundary conditions Eq.(8) gives rise to a tridiagonal set of $(n+1)$ algebraic equations which are solved by Gaussian elimination method for the $(n+1)$ unknown$y_i, i = 0, 1, 2, \dots, N$.

Nonlinear Polynomial Spline Function

The nonlinear polynomial Spline function $S(x,\tau) = S(x)$, $[x_i, x_{i+1}]$satisfying the differential equation

$$S''(x) - \tau S(x) = S''(x_i) - \tau S(x_i)\frac{(x_{i+1}-x)}{h} + S''(x_{i+1}) - \tau S(x_{i+1})\frac{(x-x_i)}{h}, \quad (18)$$

Where $S(x_i) = y_i$ and $\tau > 0$ is termed as cubic spline in tension. Solving the linear second-order differential Eq. (18) and determining the arbitrary constant from the interpolatory condition Eq.(2), $\lambda = h\tau^{\frac{1}{2}}$, we get

$$S(x) = \frac{h^2}{\lambda^2 \sin h\lambda}\left[M_{i+1} \sin h\frac{\lambda(x-x_i)}{h} + M_i \sin h\frac{\lambda(x_{i+1}-x)}{h}\right] - \frac{h^2}{\lambda^2}\left[\frac{(x-x_i)}{h}\left(M_{i+1} - \frac{\lambda^2}{h^2}y_{i+1}\right) - \frac{(x_{i+1}-x)}{h}\left(M_i - \frac{\lambda^2}{h^2}y_i\right)\right] \qquad (19)$$

Differentiating Eq.(19) and letting $x \to x_i$ i.e.one sided limit of derivatives of $S(x)$, we obtain

$$S'(x_i^+) = \frac{y_{i+1} - y_i}{h} - \frac{h}{\lambda^2}\left[\left(1 - \frac{\lambda}{\sin h\lambda}\right)M_{i+1} + (\lambda \cot h\lambda - 1)M_i\right] \qquad (20)$$

Considering the interval $[x_{i+1}, x_i]$, we obtain

$$S'(x_i^-) = \frac{y_i - y_{i-1}}{h} + \frac{h}{\lambda^2}\left[(\lambda \cot h\lambda - 1)M_i + \left(1 - \frac{\lambda}{\sin h\lambda}\right)M_{i-1}\right] \qquad (21)$$

Equating the left hand side and right hand side derivatives at $x_i, S'_{i-1}(x_i) = S'_i(x_i)$, i.e continuity condition, we have

$$\frac{y_i - y_{i-1}}{h} + \frac{h}{\lambda}\left[(\lambda \cot h\lambda - 1)M_i + \left(1 - \frac{\lambda}{\sin h\lambda}\right)M_{i+1}\right] = \frac{y_{i+1} - y_i}{h} - \frac{h}{\lambda^2}\left[\left(1 - \frac{\lambda}{\sin h\lambda}\right)M_{i+1} + (\lambda \cot h\lambda - 1)M_i\right] \qquad (22)$$

This leads to the consistency relation

$$h^2(\lambda_1 M_{i-1} + 2\lambda_2 M_i + \lambda_1 M_{i+1}) = y_{i-1} - 2y_i + y_{i+1} \qquad (23)$$

Where$\lambda_1 = \frac{1}{\lambda^2}\left(1 - \frac{\lambda}{\sin h\lambda}\right)$, $\lambda_2 = \frac{1}{\lambda^2}(\lambda \cot h\lambda - 1)$, $i = 1(1)1N - 1$.

Nonlinear Polynomial

For a numerical solution of the boundary value problem Eq.(7) the interval [0,1] is divided into a set of grid points with step length $h = \frac{(b-a)}{N}$, N being a positive integer. The spline approximation on [0, 1] that consist of the nodal point $x_0 = a$, $x_N = b$, $x_i = a + ih$, $i = 1(1)1N - 1$.

We consider two cases of Eq.(7)

Case (a): when $q(x) = 0$

Now, setting $x = x_i$, in Eq.(7)

$$-\epsilon y''(x_i) + p(x_i)y(x_i) = f(x_i), \quad (24)$$

Now, making use of Eq.(2) and substituting Eq.(9), Eq.(10) and Eq.(11) into consistency relation Eq.(23), we obtain

$$(\lambda_1 h^2 p_{i-1} - \epsilon)y_{i-1} + 2(\lambda_2 h^2 p_i + \epsilon)y_i + (\lambda_1 h^2 p_{i+1} - \epsilon)y_{i+1} = h^2(\lambda_1 f_{i-1} + 2\lambda_2 f_i + \lambda_1 f_{i+1}), \quad (25)$$

Where $i = 1(1)1N - 1$.

Thus, Eq.(25) together with the two boundary conditions Eq.(8) gives a tridiagonal system of $(n+1)$ algebraic equations which are solved by Gaussian elimination method for $(n+1)$ unknown y_i, $i = 0, 1, 2, \dots, N$.

Case (b):when $q(x) \neq 0$

Now, setting $x = x_i$ in Eq.(7) and making use of Eq.(2), Eq.(13), Eq.(14) and Eq.(15) into the consistency relation Eq.(23) and the finite difference approximation of first order derivative of y i.e Eq.(16), we have

$$\left(-\epsilon + \lambda_1 h^2 p_{i-1} - \frac{3}{2}\lambda_1 h q_{i-1} - \lambda_2 h q_i + \frac{1}{2}\lambda_1 h q_{i+1}\right) y_{i-1} + (2\epsilon + 2\lambda_2 h^2 p_i + 2\lambda_1 h q_{i-1} - 2\lambda_1 h p_{i+1}) y_i + \left(-\epsilon + \lambda_1 h^2 p_{i+1} - \frac{1}{2}\lambda_1 h q_{i-1} + \lambda_2 h q_i + \frac{3}{2}\lambda_1 h q_{i+1}\right) y_{i+1} = h^2(\lambda_1 f_{i-1} + 2\lambda_2 f_i + \lambda_1 f_{i+1}) \quad (26)$$

Where $i = 1(1)N - 1$.

Thus, Eq.(26) together with the boundary condition Eq.(8) gives rise to a tridiagonal system of $(n+1)$ algebraic algebraic equations which can be solve by Gaussian elimination method for $(n+1)$ unknown y_i , $i = 0, 1, 2, \dots, N$.

Problem (1)

$$-\epsilon y'' + 1 + x(1-x)y = 1 + x(1-x) + 2\sqrt{\epsilon} - x^2(1-x)exp^{\frac{-(1-x)}{\sqrt{\epsilon}}} + 2\sqrt{\epsilon} - x(1 - x^2)exp^{\frac{-x}{\sqrt{\epsilon}}}; \qquad y(0) = y(1) = 0.$$

The exact solution;

$$y(x) = 1 + (x-1)exp^{\left(\frac{-x}{\sqrt{\epsilon}}\right)} - xexp^{\frac{-(1-x)}{\sqrt{\epsilon}}};$$

Note:
LP = Linear Polynomial
NLP = Nonlinear Polynomial

Table 1: Problem (1) Solution $N = 16$ with $\epsilon = \frac{1}{16}$

X	Exact	Approximate LP	Error LP	Approximate NLP	Error NLP
0.00	0.0000000000	0.0000000000	$0.0000e^{+00}$	0.0000000000	$0.0000e^{+00}$
0.06	0.2684044068	0.2694052773	$1.0009e^{-03}$	0.2684083629	$3.9562e^{-06}$
0.12	0.4655109998	0.4670703220	$1.5593e^{-03}$	0.4655171911	$6.1912e^{-06}$
0.19	0.6089320119	0.6107696509	$1.8376e^{-03}$	0.6089393415	$7.3296e^{-06}$
0.25	0.7116436520	0.7135931376	$1.9495e^{-03}$	0.7116514622	$7.8102e^{-06}$
0.31	0.7830504955	0.7850229922	$1.9495e^{-03}$	0.7830584298	$7.9343e^{-06}$
0.38	0.8297617754	0.8317197029	$1.9579e^{-03}$	0.8297696770	$7.9016e^{-06}$
0.44	0.8561399961	0.8580776892	$1.9377e^{-03}$	0.8561478329	$7.8369e^{-06}$
0.50	0.8646647168	0.8665938389	$1.9291e^{-03}$	0.8646725249	$7.8081e^{-06}$
0.56	0.8561399961	0.8580776892	$1.9377e^{-03}$	0.8561478329	$7.8369e^{-06}$
0.62	0.8297617754	0.8317197029	$1.9579e^{-03}$	0.8297696770	$7.9016e^{-06}$
0.69	0.7830504955	0.7850229922	$1.9725e^{-03}$	0.7830584298	$7.9343e^{-06}$
0.75	0.7116436520	0.7135931376	$1.9495e^{-03}$	0.7116514622	$7.8102e^{-06}$
0.81	0.6089320119	0.6107696509	$1.8367e^{-03}$	0.6089393415	$7.3296e^{-06}$
0.88	0.6089320119	0.2694052773	$1.5593e^{-03}$	0.4655171911	$6.1912e^{-06}$
0.94	0.2684044068	0.2694052773	$1.0000e^{-03}$	0.2684083629	$3.9562e^{-06}$
1.00	0.0000000000	0.0000000000	$0.0000e^{+00}$	0.0000000000	$0.0000e^{+00}$

Table 2: The maximum Absolute Errors case $N = 16$

ϵ	Error LP Spline	Error NLP Spline
$\frac{1}{16}$	$1.9725e^{-03}$	$7.9343e^{-06}$
$\frac{1}{32}$	$2.9275e^{-03}$	$2.2195e^{-05}$
$\frac{1}{64}$	$5.1321e^{-03}$	$7.3085e^{-05}$
$\frac{1}{128}$	$9.4595e^{-03}$	$2.4975e^{-04}$

Table 3: The maximum Absolute Errors case $N = 32$

ϵ	Error LP Spline	Error NLP Spline
$\frac{1}{16}$	$4.9084e^{-04}$	$4.9710e^{-07}$
$\frac{1}{32}$	$7.2608e^{-04}$	$1.3933e^{-06}$
$\frac{1}{64}$	$1.2583e^{-03}$	$4.6055e^{-06}$
$\frac{1}{128}$	$2.3291e^{-03}$	$1.6298e^{-05}$

Table 4: The maximum Absolute Errors case$N = 64$

ϵ	Error LP Spline	Error NLP Spline
$\frac{1}{16}$	$1.2257e^{-04}$	$3.1100e^{-08}$
$\frac{1}{32}$	$1.8164e^{-04}$	$8.7400e^{-08}$
$\frac{1}{64}$	$3.1373e^{-04}$	$2.8880e^{-07}$
$\frac{1}{128}$	$5.8123e^{-04}$	$2.8880e^{-07}$

Problem (2)

Consider the following problem

$-\epsilon y'' + y = -\cos^2(\pi x) - 2\epsilon\pi^2 \cos(2\pi x)\,; \quad y(0) = y(1) = 0$

The exact solution: $y(x) = \frac{exp^{(\frac{x-1}{\sqrt{\epsilon}})} + exp^{\frac{-x}{\sqrt{\epsilon}}}}{1+exp^{\frac{-1}{\sqrt{\epsilon}}}} - \cos^2(\pi x)$

Table 5: Problem (2) Solution $N = 16$ with $\epsilon = \frac{1}{16}$

x	Exact	Approximate LP	Error LP	Approximate NLP	Error NLP
0.00	0.0000000000	0.0000000000	$0.0000e^{+00}$	0.0000000000	$0.0000e^{+00}$
0.06	-0.1740519068	-0.1740095234	$4.2383e^{-05}$	-0.1740585259	$6.6191e^{-06}$
0.12	-0.2282776718	-0.2288350821	$5.5741e^{-04}$	-0.2282834696	$5.7978e^{-06}$
0.19	-0.1893944408	-0.1910730187	$1.6786e^{-03}$	-0.1893937460	$6.9480e^{-07}$
0.25	-0.0898457280	-0.0929603531	$3.1146e^{-03}$	-0.0898350631	$1.0665e^{-05}$
0.31	0.0354714189	0.0308600796	$4.6113e^{-03}$	0.0354931447	$2.1726e^{-05}$
0.38	0.1532788853	0.1473690930	$5.9098e^{-03}$	0.1533104705	$3.1585e^{-05}$
0.44	0.2360916672	0.2293031586	$6.7885e^{-03}$	0.2361300087	$3.8342e^{-05}$
0.50	0.2658022288	0.2587033839	$7.0988e^{-03}$	0.2658429688	$4.0740e^{-05}$
0.56	0.2360916672	0.2293031586	$6.7885e^{-03}$	0.2361300087	$3.8342e^{-05}$
0.62	0.1532788853	0.1473690930	$5.9098e^{-03}$	0.1533104705	$3.1585e^{-05}$
0.69	0.0354714189	0.0308600796	$4.6113e^{-03}$	0.0354931447	$2.1726e^{-05}$
0.75	-0.0898457280	-0.0929603531	$3.1146e^{-03}$	-0.0898350631	$1.0665e^{-05}$
0.81	-0.1893944408	-0.1910730187	$1.6786e^{-03}$	-0.1893937460	$6.9480e^{-07}$
0.88	-0.2282776718	-0.2288350821	$5.5741e^{-04}$	-0.2282834696	$5.7978e^{-06}$
0.94	-0.1740519068	-0.1740095234	$4.2383e^{-05}$	-0.1740585259	$6.6191e^{-06}$
1.00	0.0000000000	0.0000000000	$0.0000e^{+00}$	0.0000000000	$0.0000e^{+00}$

Table 6: The maximum Absolute Errors case $N = 16$

ϵ	Error LP Spline	Error NLP Spline
$\frac{1}{16}$	$7.0988e^{-03}$	$4.0740e^{-03}$
$\frac{1}{32}$	$5.6878e^{-03}$	$2.0056e^{-05}$
$\frac{1}{64}$	$4.0736e^{-03}$	$5.4564e^{-05}$
$\frac{1}{128}$	$6.9756e^{-03}$	$1.8349e^{-04}$

Table 7: The maximum Absolute Errors case $N = 32$

ϵ	Error LP Spline	Error NLP Spline
$\frac{1}{16}$	$1.7791e^{-03}$	$2.533e^{-06}$
$\frac{1}{32}$	$1.4224e^{-03}$	$1.2422e^{-06}$
$\frac{1}{64}$	$1.0170e^{-03}$	$3.4296e^{-06}$
$\frac{1}{128}$	$1.7515e^{-03}$	$1.2261e^{-05}$

Table 8: The maximum Absolute Errors case $N = 64$

ϵ	Error LP Spline	Error NLP Spline
$\frac{1}{16}$	$4.4506e^{-04}$	$1.5810e^{-07}$
$\frac{1}{32}$	$3.5563e^{-04}$	$7.7300e^{-08}$
$\frac{1}{64}$	$2.5415e^{-04}$	$2.1450e^{-07}$
$\frac{1}{128}$	$4.3394e^{-04}$	$7.6920e^{-07}$

Table 9: The maximum Absolute Errors case $N = 128$

ϵ	Error LP Spline	Error NLP Spline
$\frac{1}{16}$	$1.1128e^{-04}$	$9.9000e^{-09}$
$\frac{1}{32}$	$8.8910e^{-05}$	$4.7000e^{-09}$
$\frac{1}{64}$	$6.3533e^{-05}$	$1.3400e^{-08}$
$\frac{1}{128}$	$1.0853e^{-04}$	$4.8300e^{-08}$

RESULT DISCUSSION

Both Problem (1) & (2) were solved by both Linear polynomial method and Nonlinear polynomial method and different values of N=16, 32 and 64 with different values of $\epsilon = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$ and $\frac{1}{128}$. The maximum absolute error defined by Absolute Error= $\max_{a \le x \le b} |\bar{y}_i - y_i|$,$\bar{y}_i$is the exact solution and $y_i$is the approximate solution. The numerical solution and the errors for both problems (1) and (2) were presented and later compare the maximum absolute errors with different values of N and ϵ between the two methods.

CONCLUSION

We solved some singularly perturbed boundary value problems and the results obtained are compared between the two methods. The results presented are linear polynomial and nonlinear polynomial splines together with their maximum absolute errors, which show that the nonlinear polynomial Spline is better than the linear polynomial Spline in-terms of accuracy as compared with the exact solution.

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